# Manifold Learning for Signal and Image Analysis
# Lecture 6: Graph and Diffusion Kernels

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# Outline of Lecture 6

- Kernels on graphs
- The exponential diffusion kernel
- The heat kernel of discrete manifolds
- Properties of the heat kernel
- The auto-diffusion function
- Shape matching

# Material for this lecture

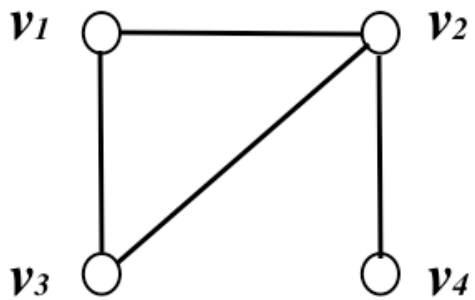
- P. Bérard, G. Besson, & G. Gallot. Embedding Riemannian manifolds by their heat kernel. Geometric and Functional Analysis (1994): A rather theoretical paper.
- J. Shawe-Taylor & N. Cristianini. Kernel Methods in Pattern Analysis (chapter 10): A mild introduction to graph kernels.
- R. Kondor and J.-P. Vert: Diffusion Kernels in "Kernel Methods in Computational Biology" ed. B. Scholkopf, K. Tsuda and J.-P. Vert, (The MIT Press, 2004): An interesting paper to read.
- A. Sharma & R. Horaud. Shape matching based on diffusion embedding and on mutual isometric consistency. CVPR-NORDIA workshop (2010).

# The Adjacency Matrix of a Graph (from Lecture #3)

- For a graph with $n$ vertices and with binary edges, the entries of the $n \times n$ adjacency matrix are defined by:

$$\mathbf{A} := \begin{cases} A_{ij} = 1 & \text{if there is an edge } e_{ij} \\ A_{ij} = 0 & \text{if there is no edge} \\ A_{ii} = 0 & \end{cases}$$

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



# Powers and Exponential of a Symmetric Matrix

- $\mathbf{A}^2 = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top = \mathbf{U}\mathbf{\Lambda}^2\mathbf{U}^\top$
- More generally: $\mathbf{A}^k = \mathbf{U}\mathbf{\Lambda}^k\mathbf{U}^\top$
- Matrices $\mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^k$ have the same eigenvectors $\{\boldsymbol{u}_i\}$;
- Their eigenvalues are $\lambda, \lambda^2, \ldots, \lambda^k$.
- Matrix exponential: $e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$
- We have: $e^{\mathbf{A}} = \mathbf{U}\text{Diag}[e^{\lambda_1} \ldots e^{\lambda_i} \ldots e^{\lambda_D}]\mathbf{U}^\top$
- Hence, matrix $e^{\mathbf{A}}$ has the same eigenvectors as $\mathbf{A}$ and eigenvalues $e^{\lambda}$.

# The Walks of Length 2

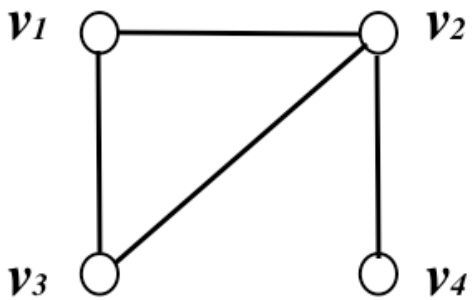
- Notation: $\mathbf{A}^k(i,j)$ is the $(i,j)$ entry of $\mathbf{A}^k$.
- The number of walks of length 2 between two graph vertices:

$$N_2(v_i, v_j) = A^2(i,j)$$

- Examples:

$$N_2(v_2, v_3) = 1; \quad N_2(v_2, v_2) = 3$$

$$\mathbf{A}^2 = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 0 \\ 1 & 1 & 2 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$



# The Number of Walks of Length $k$

**Theorem**

*The number of walks of length $k$ joining any two nodes $v_i$ and $v_j$ of a binary graph is given by the $(i, j)$ entry of the matrix $\mathbf{A}^k$:*

$$N_k(v_i, v_j) = \mathbf{A}^k(i, j)$$

- A proof of this theorem is provided in:
  Godsil and Royle. (2001). Algebraic Graph Theory. Springer.

# The Similarity Matrix of an Undirected Weighted Graph

- We consider *undirected weighted graphs*; Each edge $e_{ij}$ is weighted with $\omega_{ij} > 0$. We obtain:

$$\mathbf{\Omega} := \begin{cases} \Omega(i,j) = \omega(v_i, v_j) = \omega_{ij} & \text{if there is an edge } e_{ij} \\ \Omega(i,j) = 0 & \text{if there is no edge} \\ \Omega(i,i) = 0 \end{cases}$$

- We will refer to this matrix as the *base* similarity matrix

# Walks of Length $k$

- **Definition:** The weight of a walk from node $v_i$ to node $v_j$ through node $v_l$ is the *product* of the corresponding edge weights: $\omega(v_i, v_j) = \omega_{il}\omega_{lj}$
- The $(i, j)$ entry of the squared weighted adjacency matrix:

$$\mathbf{\Omega}^2(i, j) = \sum_{l=1}^{n} \omega_{il}\omega_{lj}$$

- This can be interpreted as the sum of the walks of length 2:

$$\omega_2(v_i, v_j) = \mathbf{\Omega}^2(i, j)$$

- More generally, the sum of the walks of length $k$ is:

$$\omega_k(v_i, v_j) = \mathbf{\Omega}^k(i, j)$$

# Other "Adjacency" Matrices

- The *normalized weighted adjacency matrix*

$$\mathbf{\Omega}_N = \mathbf{D}^{-1/2}\mathbf{\Omega}\mathbf{D}^{-1/2}$$

- The *transition* matrix of the Markov process associated with the graph:

$$\mathbf{\Omega}_R = \mathbf{D}^{-1}\mathbf{\Omega} = \mathbf{D}^{-1/2}\mathbf{\Omega}_N\mathbf{D}^{1/2}$$

- Notice that it is a row-stochastic matrix, namely:

$$\sum_j \mathbf{\Omega}_R(i,j) = 1$$

# Random Walks

- Starting from any node $v$, select at random one of its neighbors and move to this node, etc. The sequence of randomly selected nodes is a **random walk**.
- $p_{ij}^1 = \mathbf{\Omega}_R(i,j) = \omega_{ij}/d_i$ is the probability of *transition in one step* from node $v_i$ to node $v_j$.

**Theorem**

*The entry $\mathbf{\Omega}_R^k(i,j)$ gives the probability $p_{ij}^k$ of transition from node $v_i$ to node $v_j$ in $k$ steps.*

# Interpretation

- Powers of the adjacency matrices can serve as a similarity measure between any two nodes.
- The power of an adjacency matrix of a graph is the adjacency matrix of a *new* graph that incorporates the local structure of the initial graph in the node-similarity measure.
- The power $k$ defines the *scale* at which the local structure is considered.

# Back to Kernel Methods: Associated Feature Space

- Let $\mathbf{\Omega} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$, i.e., the spectral decomposition of a real symmetric matrix.
- We have $\mathbf{\Omega}^2 = \mathbf{U}\mathbf{\Lambda}^2\mathbf{U}^\top$ is symmetric semi-definite *positive*, hence it can be interpreted as a kernel matrix, with entries:

$$\begin{aligned} \omega_2(v_i, v_j) &= \sum_{l=1}^{n} \omega_{il}\omega_{lj} \\ &= \left\langle (\omega_{il})_{l=1}^{n}, (\omega_{jl})_{l=1}^{n} \right\rangle = \kappa(v_i, v_j) \end{aligned}$$

- Associated feature space:

$$\phi : v_i \to \boldsymbol{\phi}(v_i) = (\omega_{i1} \ldots \omega_{il} \ldots \omega_{in})^\top$$

- It is possible to "enhance" the base similarity matrix by linking vertices along walks of length 2.

# Combining Powers of the Base Similarity Matrix

- More generally one can take powers of the base similarity matrix as well as linear combinations of these matrices:

$$\alpha_1 \mathbf{\Omega} + \ldots + \alpha_k \mathbf{\Omega}^k + \ldots + \alpha_K \mathbf{\Omega}^K$$

- The eigenvalues of such a matrix must be nonnegative to satisfy the kernel condition:

$$\alpha_1 \mathbf{\Lambda} + \ldots + \alpha_k \mathbf{\Lambda}^k + \ldots + \alpha_K \mathbf{\Lambda}^K \succeq 0$$

# The Exponential Diffusion Kernel

- Consider the following combination of powers of the base similarity matrix:

$$\mathbf{K} = \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \mathbf{\Omega}^k$$

- This corresponds to:

$$\mathbf{K} = e^{\mu \mathbf{\Omega}}$$

- where $\mu$ is a *decay factor* chosen such that the influence of longer walks decreases, since they are less reliable.
- Spectral decomposition:

$$\mathbf{K} = \mathbf{U} e^{\mu \mathbf{\Lambda}} \mathbf{U}^\top$$

- Hence $\mathbf{K}$ is a kernel matrix since its eigenvalues $e^{\mu \lambda_i} \geq 0$.

# Heat Diffusion on a Graph

- The heat-diffusion equation on a Riemannian manifold: $\left(\frac{\partial}{\partial t} + \Delta_{\mathcal{M}}\right) f(x;t) = 0$
- $\Delta_{\mathcal{M}}$ denotes the geometric Laplace-Beltrami operator.
- $f(x;t)$ is the distribution of heat over the manifold at time $t$.
- By extension, $\frac{\partial}{\partial t} + \Delta_{\mathcal{M}}$ can be referred to as the *heat operator* [Bérard et al. 1994].
- This equation can also be written on a graph:

$$\left(\frac{\partial}{\partial t} + \mathbf{L}\right) \boldsymbol{F}(t) = 0$$

where the vector $\boldsymbol{F}(t) = (F_1(t) \dots F_n(t))^\top$ is indexed by the nodes of the graph.

# The Fundamental Solution

- The fundamental solution of *the (heat)-diffusion equation on Riemannian manifolds* holds in the discrete case, i.e., for undirected weighted graphs.
- The solution in the discrete case is:

$$\boldsymbol{F}(t) = \mathbf{H}(t)\boldsymbol{f}$$

- where $\mathbf{H}$ denotes the discrete heat operator:

$$\mathbf{H}(t) = e^{-t\mathbf{L}}$$

- $\boldsymbol{f}$ corresponds to the initial heat distribution:

$$\boldsymbol{F}(0) = \boldsymbol{f}$$

- Starting with a point-heat distribution at vertex $v_i$, e.g., $(0 \dots f_i = 1 \dots 0)^\top$, the distribution at $t$, i.e., $\boldsymbol{F}(t) = (F_1(t) \dots F_n(t))$ is given by the $i$-th column of the heat operator.

# How to Compute the Heat Matrix?

- The exponential of a matrix:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!}$$

- Hence:

$$e^{-t\mathbf{L}} = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \mathbf{L}^k$$

- It belongs to the "exponential diffusion" family of kernels just introduced with $\mu = -t, t > 0$.

# Spectral Properties of $\mathbf{L}$

We start by recalling some basic facts about the combinatorial graph Laplacian:

- Symmetric semi-definite positive matrix: $\mathbf{L} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top$
- Eigenvalues: $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$
- Eigenvectors: $\boldsymbol{u}_1 = \mathbb{1}, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$
- $\lambda_2$ and $\boldsymbol{u}_2$ are the Fiedler value and the Fiedler vector
- $\boldsymbol{u}_i^\top \boldsymbol{u}_j = \delta_{ij}$
- $\boldsymbol{u}_{i>1}^\top \mathbb{1} = 0$
- $\sum_{i=1}^n u_{ik} = 0, \forall k \in \{2, \ldots, n\}$
- $-1 < u_{ik} < 1, \forall i \in \{1, \ldots, n\}, \forall k \in \{2, \ldots, n\}$

$$\mathbf{L} = \sum_{k=2}^{n} \lambda_k \boldsymbol{u}_k \boldsymbol{u}_k^\top$$

# The Heat-kernel Matrix

$$\mathbf{H}(t) = e^{-t\mathbf{U}\mathbf{\Lambda}\mathbf{U}^\top} = \mathbf{U}e^{-t\mathbf{\Lambda}}\mathbf{U}^\top \succeq 0$$

with:

$$e^{-t\mathbf{\Lambda}} = \mathsf{Diag}[e^{-t\lambda_1} \dots e^{-t\lambda_n}]$$

- Eigenvalues: $1 = e^{-t0} > e^{-t\lambda_2} \geq \ldots \geq e^{-t\lambda_n}$
- Eigenvectors: same as the Laplacian matrix with their properties (previous slide).
- The heat trace (also referred to as the partition function):

$$Z(t) = \mathsf{tr}(\mathbf{H}) = \sum_{k=1}^{n} e^{-t\lambda_k}$$

- The determinant:

$$\det(\mathbf{H}) = \prod_{k=1}^{n} e^{-t\lambda_k} = e^{-t\mathsf{tr}(\mathbf{L})} = e^{-t\mathsf{vol}(\mathcal{G})}$$

# The Heat-kernel

- Computing the heat matrix:

$$\mathbf{H}(t) = \sum_{k=2}^{n} e^{-t\lambda_k} \boldsymbol{u}_k \boldsymbol{u}_k^\top$$

  where we applied a *deflation* to get rid of the constant eigenvector: $\mathbf{H} \longrightarrow \mathbf{H} - \boldsymbol{u}_1 \boldsymbol{u}_1^\top$

- The heat kernel (en entry of the matrix above):

$$h(i, j; t) = \sum_{k=2}^{n} e^{-t\lambda_k} u_{ik} u_{jk}$$

# Feature-space Embedding Using the Heat Kernel

$$\mathbf{H}(t) = \left(\mathbf{U}e^{-\frac{1}{2}t\mathbf{\Lambda}}\right)\left(\mathbf{U}e^{-\frac{1}{2}t\mathbf{\Lambda}}\right)^{\top}$$

- Each row of the $n \times n$ matrix $\mathbf{U}e^{-t\mathbf{\Lambda}/2}$ can be viewed as the coordinates of a graph vertex in a feature space, i.e., the mapping $\phi : \mathcal{V} \to \mathbb{R}^{n-1}$, $\boldsymbol{x}_i = \boldsymbol{\phi}(v_i)$:

$$\begin{aligned}\boldsymbol{x}_i &= \left( e^{-\frac{1}{2}t\lambda_2}u_{i2} \quad \dots \quad e^{-\frac{1}{2}t\lambda_k}u_{ik} \quad \dots \quad e^{-\frac{1}{2}t\lambda_n}u_{in} \right)^{\top} \\ &= (x_{i2} \dots x_{ik} \dots x_{in})^{\top}\end{aligned}$$

- The heat-kernel computes the inner product in feature space:

$$h(i, j; t) = \langle \boldsymbol{\phi}(v_i), \boldsymbol{\phi}(v_j) \rangle$$

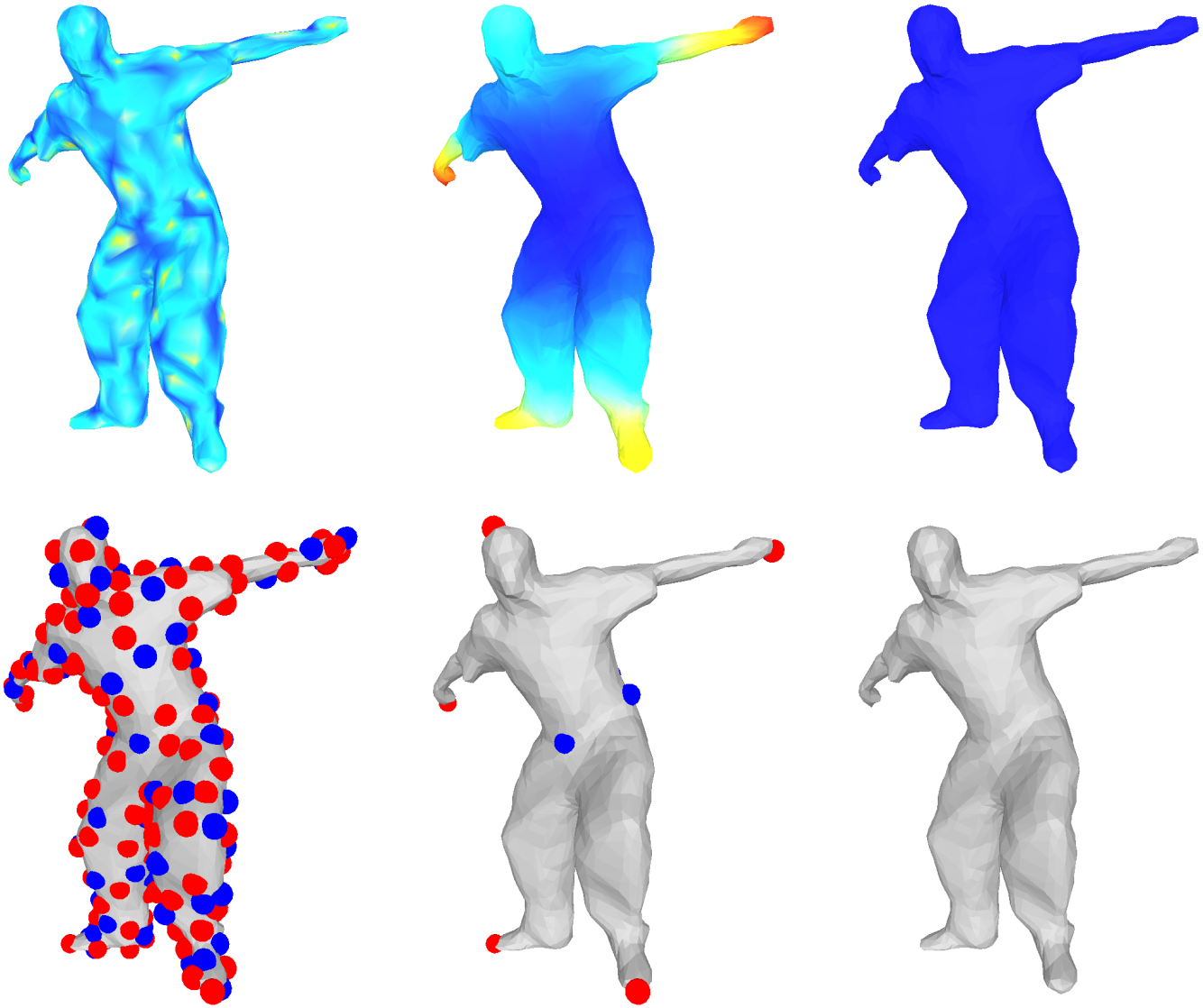
# The Auto-diffusion Function

- Each diagonal term of the heat matrix corresponds to the norm of a feature-space point:

$$h(i,i;t) = \sum_{k=2}^{n} e^{-t\lambda_k} u_{ik}^2 = \|\boldsymbol{x}_i\|^2$$

- This is also known as the *auto-diffusion function* (ADF), or the amount of heat that remains at a vertex at time $t$.
- The local maxima/minima of this function have been used for a feature-based scale-space representation of shapes.
- Associated shape descriptor: $v_i \longrightarrow h(i,i;t)$ hence it is a scalar function defined over the graph.

# The ADF as a Shape Descriptor



$t = 0$ $t = 50$ $t = 500000$

# Spectral Distances

- The heat distance:

$$\begin{aligned} d_t^2(i,j) &= h(i,i;t) + h(j,j;t) - 2h(i,j;t) \\ &= \sum_{k=2}^{n} (e^{-\frac{1}{2}t\lambda_k}(u_{ik} - u_{jk}))^2 \end{aligned}$$

- The commute-time distance:

$$\begin{aligned} d_{\mathsf{CTD}}^2(i,j) &= \int_{t=0}^{\infty} \sum_{k=2}^{n} (e^{-\frac{1}{2}t\lambda_k}(u_{ik} - u_{jk}))^2 dt \\ &= \sum_{k=2}^{n} \left(\lambda_k^{-1/2}(u_{ik} - u_{jk})\right)^2 \end{aligned}$$

# Principal Component Analysis

- The covariance matrix in feature space:

$$\mathbf{C}_X = \frac{1}{n}\sum_{i=1}^{n}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top$$

- With:

$$\mathbf{X} = \left(\mathbf{U}e^{-\frac{1}{2}t\boldsymbol{\Lambda}}\right)^\top = [\boldsymbol{x}_1 \dots \boldsymbol{x}_i \dots \boldsymbol{x}_n]$$

- Remember that each column of $\mathbf{U}$ sums to zero.
- $-1 < -e^{-\frac{1}{2}t\lambda_k} < x_{ik} < e^{-\frac{1}{2}t\lambda_k} < 1, \forall 2 \leq k \leq n$

# Principal Component Analysis: The Mean

$$
\begin{aligned}
\overline{\boldsymbol{x}} &= \frac{1}{n}\sum_{i=1}^{n}\boldsymbol{x}_i \\
&= \frac{1}{n}e^{-\frac{1}{2}t\boldsymbol{\Lambda}}\begin{pmatrix} \sum_{i=1}^{n} u_{i2} \\ \vdots \\ \sum_{i=1}^{n} u_{in} \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}
\end{aligned}
$$

# Principal Component Analysis: The Covariance

$$
\begin{aligned}
\mathbf{C}_X &= \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i \boldsymbol{x}_i^\top \\
&= \frac{1}{n}\mathbf{X}\mathbf{X}^\top \\
&= \frac{1}{n}\left(\mathbf{U}e^{-\frac{1}{2}t\boldsymbol{\Lambda}}\right)^\top \left(\mathbf{U}e^{-\frac{1}{2}t\boldsymbol{\Lambda}}\right) \\
&= \frac{1}{n}e^{-t\boldsymbol{\Lambda}}
\end{aligned}
$$

# Result I: The PCA of a Graph

- The eigenvectors (of the combinatorial Laplacian) are the principal components of the heat-kernel embedding: hence we obtain a maximum-variance embedding
- The associated variances are $e^{-t\lambda_2}/n, \ldots, e^{-t\lambda_n}/n$.
- The embedded points are strictly contained in a hyper-parallelepipedon with volume $\prod_{i=2}^{n} e^{-t\lambda_i}$.

# Dimensionality Reduction (1)

- Dimensionality reduction consists in selecting the $K$ largest eigenvalues, $K < n$, conditioned by $t$, hence the criterion: choose $K$ and $t$, such that (scree diagram):

$$\alpha(K) = \frac{\sum_{i=2}^{K+1} e^{-t\lambda_i}/n}{\sum_{i=2}^{n} e^{-t\lambda_i}/n} \approx 0.95$$

- This is not practical because one needs to compute all the eigenvalues.

# Dimensionality Reduction (2)

- An alternative possibility is to use the determinant of the covariance matrix, and to choose the first $K$ eigenvectors such that (with $\alpha > 1$):

$$\alpha(K) = \ln \frac{\prod_{i=2}^{K+1} e^{-t\lambda_i}/n}{\prod_{i=2}^{n} e^{-t\lambda_i}/n}$$

- which yields:

$$\alpha(K) = t \left( \text{tr}(\mathbf{L}) - \sum_{i=2}^{K+1} \lambda_i \right) + (n - K) \ln n$$

- This allows to choose $K$ for a scale $t$.

# Normalizing the Feature-space

Observe that the heat-kernels collapse to $0$ at infinity: $\lim_{t\to\infty} h(i,j;t) = 0$. To prevent this problem, several normalizations are possible:

- Trace normalization
- Unit hyper-sphere normalization
- Time-invariant embedding

# Trace Normalization

- Observe that $\lim_{t\to\infty} h(i,j;t) = 0$
- Use the trace of the operator to normalize the embedding:

$$\widehat{\boldsymbol{x}}_i = \frac{\boldsymbol{x}_i}{\sqrt{Z(t)}}$$

with: $Z(t) \approx \sum_{k=2}^{K+1} e^{-t\lambda_k}$

- the $k$-component of the $i$-coordinate writes:

$$\hat{x}_{ik}(t) = \frac{\left(e^{-t\lambda_k} u_{ik}^2\right)^{1/2}}{\left(\sum_{l=2}^{K+1} e^{-t\lambda_l}\right)^{1/2}}$$

- At the limit:

$$\widehat{\boldsymbol{x}}_i(t\to\infty) = \left( \begin{array}{ccccccc} \frac{u_{i2}}{\sqrt{m}} & \dots & \frac{u_{i\,m+1}}{\sqrt{m}} & 0 & \dots & 0 \end{array} \right)^\top$$

where $m$ is the multiplicity of the first non-null eigenvalue.

# Unit Hyper-sphere Normalization

- The embedding lies on a unit hyper-sphere of dimension $K$:

$$\widetilde{\boldsymbol{x}}_i = \frac{\boldsymbol{x}_i}{\|\boldsymbol{x}_i\|}$$

- The heat distance becomes a geodesic distance on a spherical manifold:

$$d_{\mathcal{S}}(i,j;t) = \arccos \widetilde{\boldsymbol{x}}_i^\top \widetilde{\boldsymbol{x}}_j = \arccos \frac{h(i,j;t)}{(h(i,i;t)h(j,j;t))^{1/2}}$$

- At the limit ($m$ is the multiplicity of the largest non-null eigenvalue):

$$\widetilde{\boldsymbol{x}}_i(t \to \infty) = \left( \begin{array}{ccccccc} \frac{u_{i2}}{\left(\sum_{l=2}^{m+1} u_{il}^2\right)^{1/2}} & \cdots & \frac{u_{i\,m+1}}{\left(\sum_{l=2}^{m+1} u_{il}^2\right)^{1/2}} & 0 & \cdots & 0 \end{array} \right)^\top$$

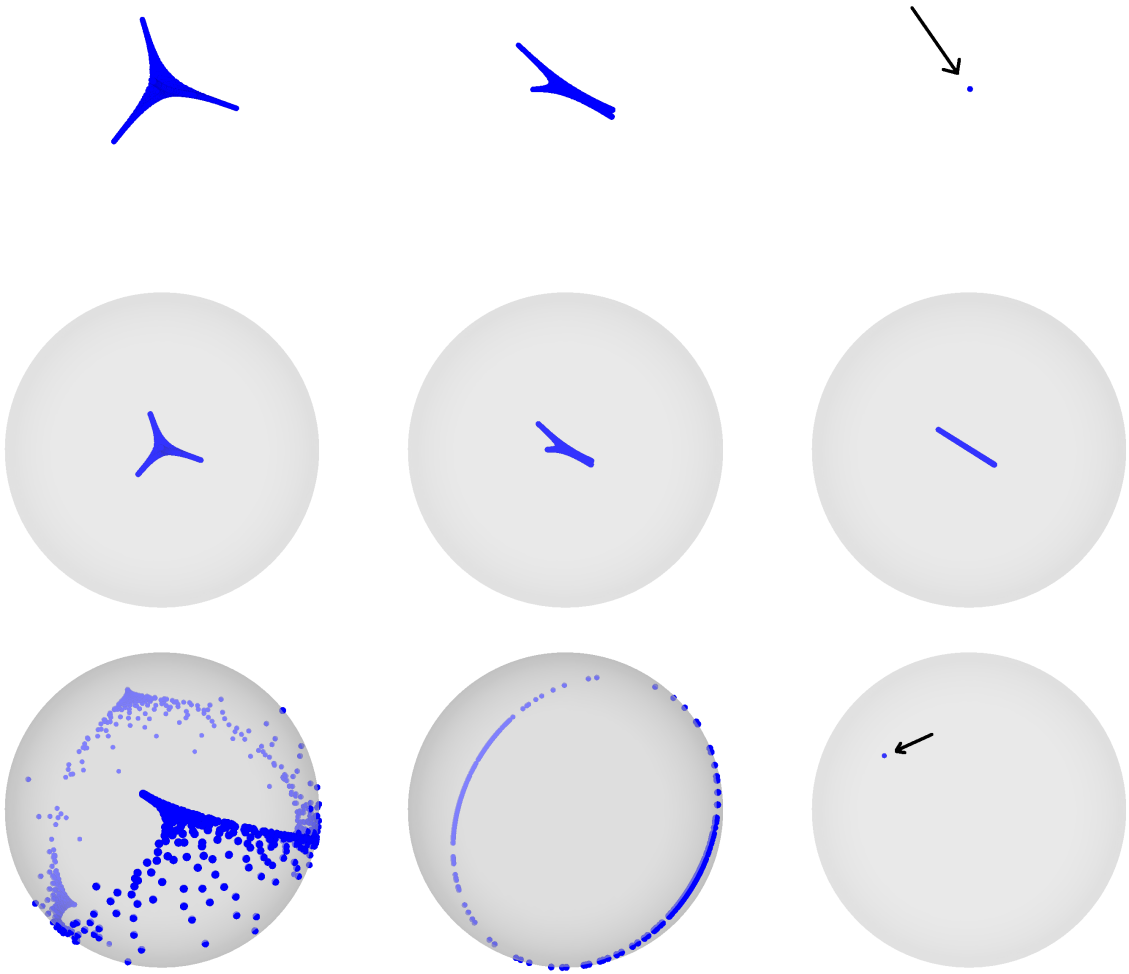
# Time-invariant Embedding

- Integration over time:

$$\begin{aligned}\mathbf{L}^{\dagger} &= \int_0^{\infty} \mathbf{H}(t) = \int_0^{\infty} \sum_{k=2}^{n} e^{-t\lambda_k} \boldsymbol{u}_k \boldsymbol{u}_k^{\top} dt \\ &= \sum_{k=2}^{n} \frac{1}{\lambda_k} \boldsymbol{u}_k \boldsymbol{u}_k^{\top} = \mathbf{U} \boldsymbol{\Lambda}^{\dagger} \mathbf{U}^{\top}\end{aligned}$$

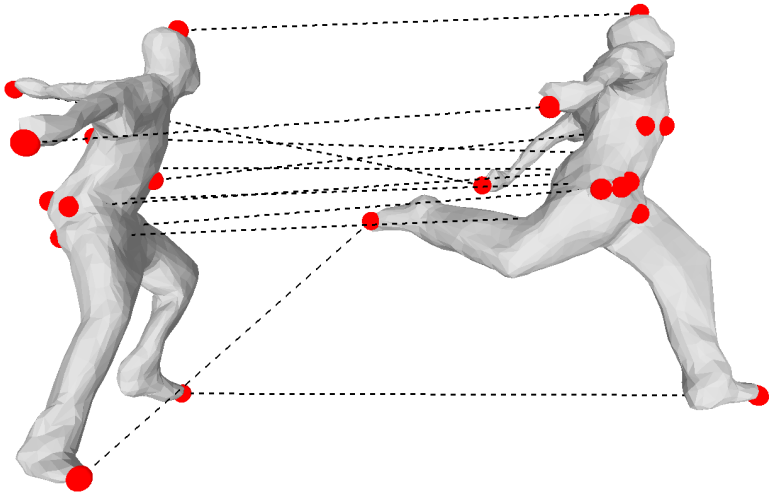
- with: $\boldsymbol{\Lambda}^{\dagger} = \text{Diag}[\lambda_2^{-1}, \ldots, \lambda_n^{-1}]$.
- Matrix $\mathbf{L}^{\dagger}$ is called the *discrete Green's function* [ChungYau2000], the Moore-Penrose pseudo-inverse of the Laplacian.
- Embedding: $\boldsymbol{x}_i = \left( \lambda_2^{-1/2} u_{i2} \ \ldots \ \lambda_{K+1}^{-1/2} u_{i\,K+1} \right)^{\top}$
- Covariance: $\mathbf{C}_X = \frac{1}{n} \text{Diag}[\lambda_2^{-1}, \ldots, \lambda_{K+1}^{-1}]$

# Examples of Normalized Embeddings

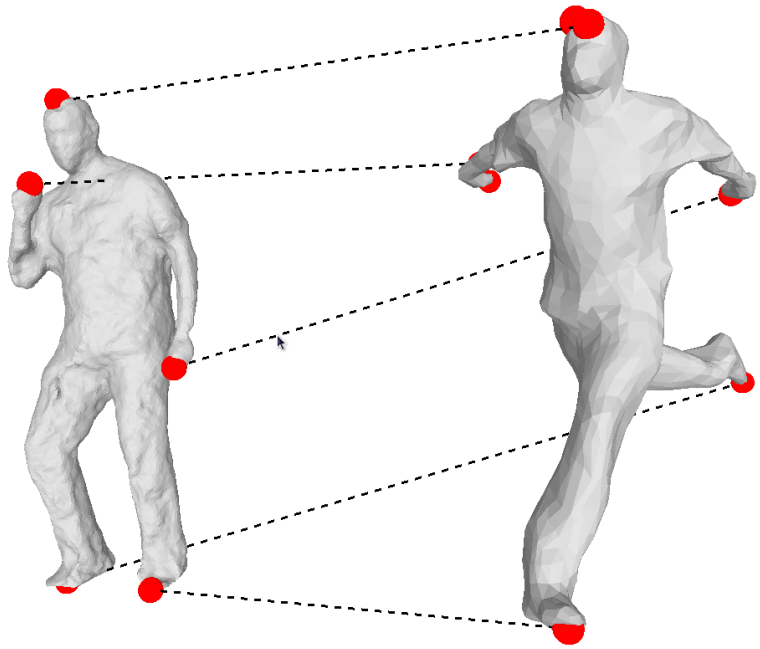


$t = 50$ $t = 5000$ $t = 500000$

# Shape Matching (1)

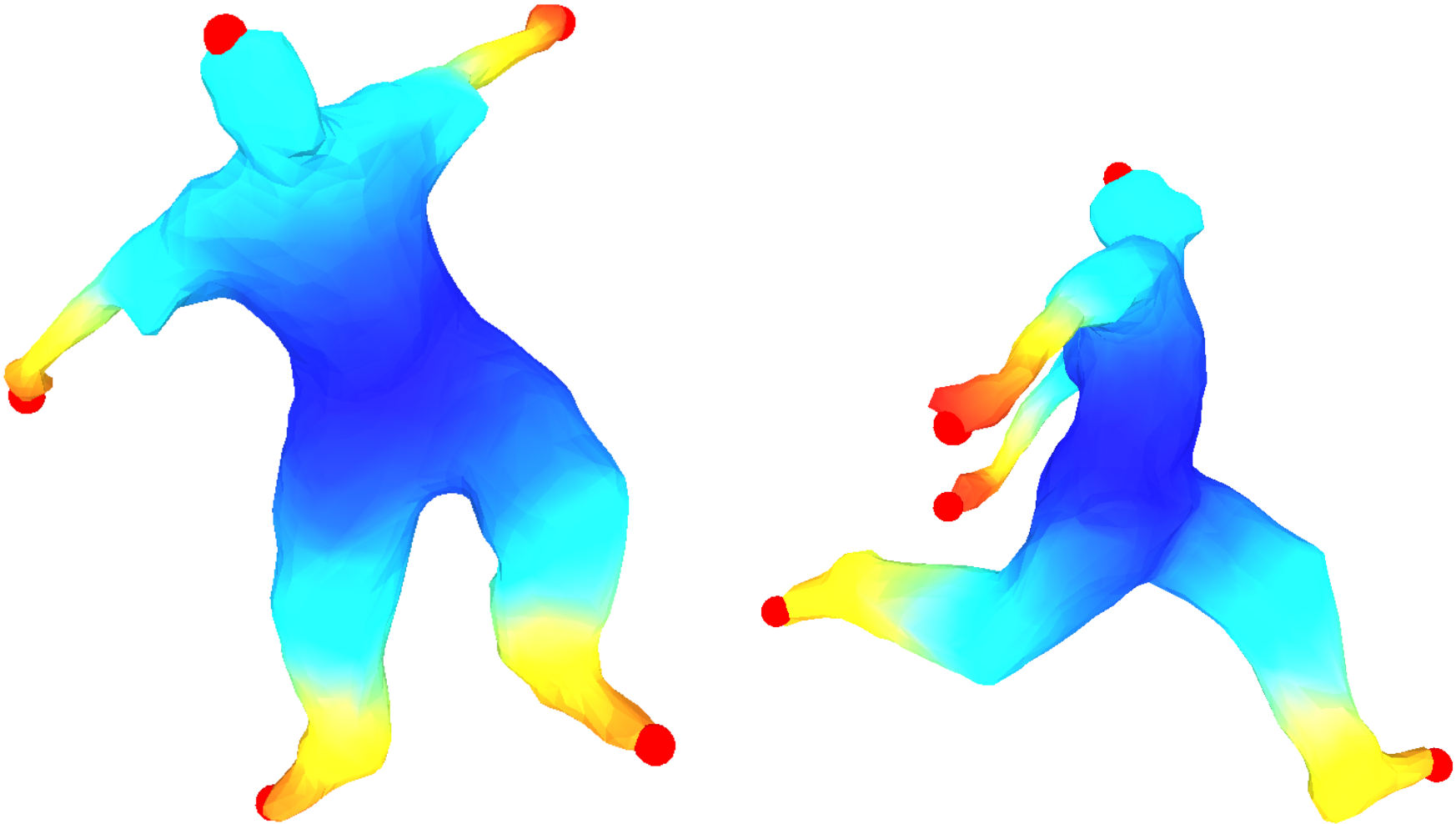


$t = 200,\ t' = 201.5$

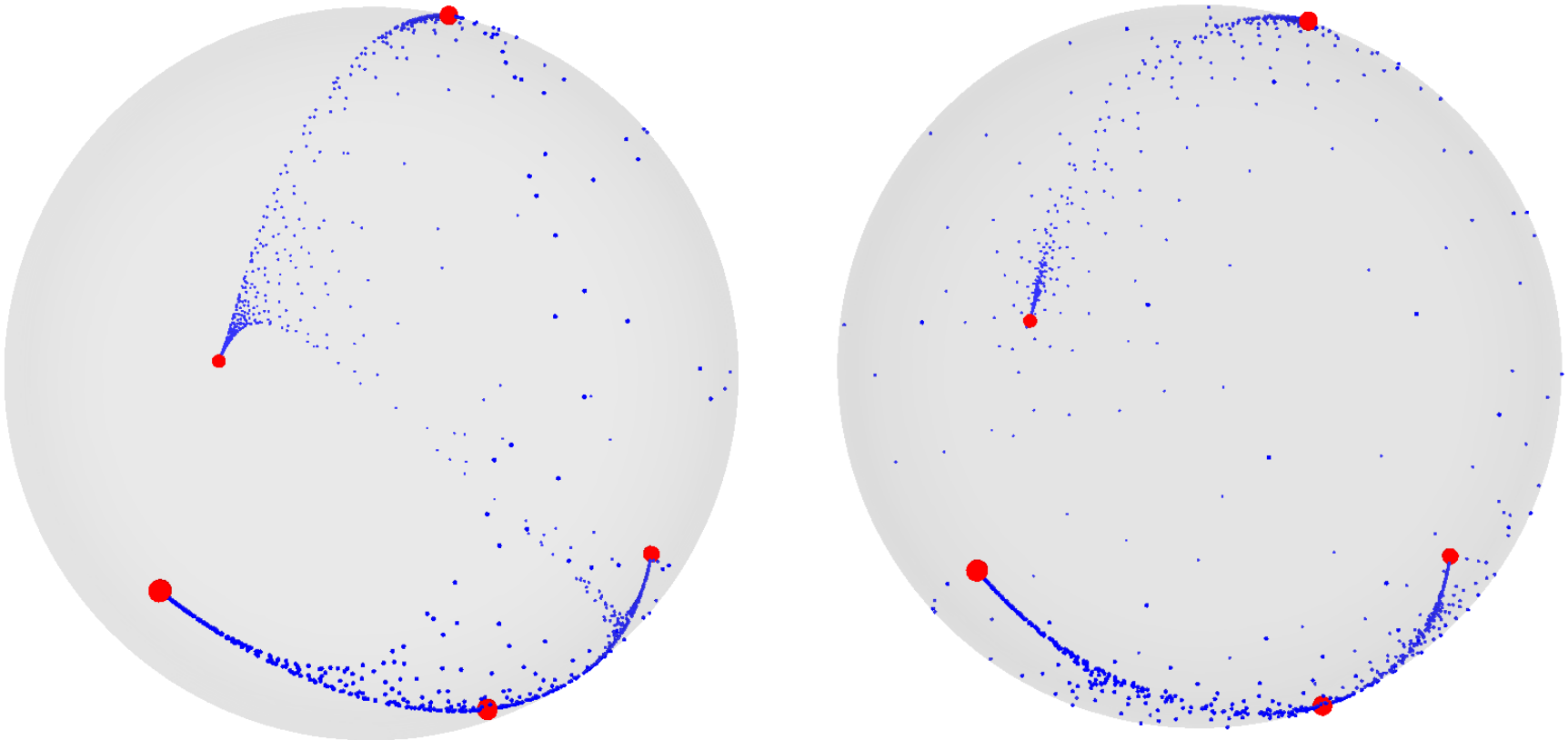


$t = 90,\ t' = 1005$

# Shape Matching (2)



# Shape Matching (3)



# Sparse Shape Matching

- Shape/graph matching is equivalent to matching the embedded representations [Mateus et al. 2008]
- Here we use the projection of the embeddings on a unit hyper-sphere of dimension $K$ and we apply rigid matching.
- How to select $t$ and $t'$, i.e., the scales associated with the two shapes to be matched?
- How to implement a robust matching method?

# Scale Selection

- Let $\mathbf{C}_X$ and $\mathbf{C}_{X'}$ be the covariance matrices of two different embeddings $\mathbf{X}$ and $\mathbf{X}'$ with respectively $n$ and $n'$ points:

$$\det(\mathbf{C}_X) = \det(\mathbf{C}_{X'})$$

- $\det(\mathbf{C}_X$ measures the volume in which the embedding $X$ lies. Hence, we impose that the two embeddings are contained in the same volume.
- From this constraint we derive:

$$t' \operatorname{tr}(\mathbf{L}') = t \operatorname{tr}(\mathbf{L}) + K \log n/n'$$

# Robust Matching

- Build an association graph.
- Search for the largest set of mutually compatible nodes (maximal clique finding).
- See [Sharma and Horaud 2010] (Nordia workshop) for more details.