

Consonant approximations of belief functions on the consonant simplex

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Abstract

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1 The consonant approximation problem

Given a belief function $b \in \mathcal{B}$ and a sub-class of b.f.s \mathcal{A} , the problem consists on finding the function $a \in \mathcal{A}$ which the “closest” to b , according to some criterion (for instance by minimizing a certain distance function $d(a, b)$).

1.1 Previous solutions

1.2 Paper outline

2 Geometry of consonant belief functions

2.1 Consonant belief functions

The *theory of evidence* [1] was introduced in the late Seventies by Glenn Shafer as a way of representing epistemic knowledge, starting from a sequence of

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seminal works [2–4], of Arthur Dempster. In this formalism the most suitable representation of subjective probability is a *belief function* (b.f.) rather than a Bayesian mass distribution, assigning probability values to *sets* of possibilities rather than single events.

Definition 1. A basic probability assignment (*b.p.a.*) over a finite set (frame of discernment [1]) Θ is a function $m : 2^\Theta \rightarrow [0, 1]$ on its power set $2^\Theta = \{A \subseteq \Theta\}$ such that $m(\emptyset) = 0$, $\sum_{A \subseteq \Theta} m(A) = 1$, and $m(A) \geq 0 \forall A \subseteq \Theta$.

Subsets of Θ associated with non-zero values of m are called *focal elements*.

Definition 2. The belief function $b : 2^\Theta \rightarrow [0, 1]$ associated with a basic probability assignment m on Θ is defined as: $b(A) = \sum_{B \subseteq A} m(B)$.

Conversely, the unique basic probability assignment m_b associated with a given belief function b can be recovered by means of the *Moebius inversion formula*¹

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B) \quad (1)$$

so that there is a 1-1 correspondence between the two set functions $m_b \leftrightarrow b$. A dual mathematical representation of the evidence encoded by a belief function b is the *plausibility function* (pl.f.) $pl_b : 2^\Theta \rightarrow [0, 1]$, $A \mapsto pl_b(A)$ where the plausibility value $pl_b(A)$ of an event A is given by

$$pl_b(A) \doteq 1 - b(A^c) = 1 - \sum_{B \subseteq A^c} m_b(B) = \sum_{B \cap A \neq \emptyset} m_b(B) \geq b(A) \quad (2)$$

and expresses the amount of evidence *not against* A . Belief functions admit the following order relation

$$b \leq b' \equiv b(A) \leq b'(A) \quad \forall A \subseteq \Theta \quad (3)$$

called *weak inclusion*.

In the theory of evidence a probability function is simply a peculiar belief function assigning non-zero masses to singletons only (*Bayesian b.f.*): $m_b(A) = 0 \mid A \mid > 1$. At the opposite of Bayesian b.f.s stand *consonant* belief functions.

Definition 3. A b.f. is said to be *consonant* if its focal elements are nested.

Proposition 1 [1] illustrates some of their properties.

Proposition 1. If b is a belief function with plausibility function pl_b , then the following conditions are equivalent:

- (1) b is consonant;
- (2) $b(A \cap B) = \min(b(A), b(B))$ for every $A, B \subseteq \Theta$;
- (3) $pl_b(A \cup B) = \max(pl_b(A), pl_b(B))$ for every $A, B \subseteq \Theta$;
- (4) $pl_b(A) = \max_{x \in A} pl_b(x)$ for all non-empty $A \subseteq \Theta$.

¹ See [5] for an explanation in terms of the theory of monotone functions on partially ordered sets.

2.2 Geometric approach to uncertainty

Motivated by the search for meaningful probabilistic approximations of belief functions we introduced the notion of *belief space* ([6], [7], [8]), as the space of all b.f.s we can define on a given domain. The approximation problem can then be posed in this framework by asking where probability functions live in this space, and which is the correct distance to use to evaluate the difference between a belief function and a probability.

More precisely, consider a frame of discernment Θ and introduce in the Cartesian space \mathbb{R}^{N-1} , $N = 2^{|\Theta|}$ an orthonormal reference frame $\{X_A : \emptyset \subsetneq A \subseteq \Theta\}$ (note that \emptyset is not included). Each vector $v = \sum_{\emptyset \subsetneq A \subseteq \Theta} v_A X(A)$ in \mathbb{R}^{N-1} is then potentially a b.f., each component v_A measuring the belief value of A : $v_A = b(A)$. Not every such vector $v \in \mathbb{R}^{N-1}$, though, represents a valid b.f.

Definition 4. *The belief space associated with Θ is the set of points \mathcal{B}_Θ of \mathbb{R}^{N-1} corresponding to a belief function.*

Assuming the domain Θ fixed we can denote the belief space by \mathcal{B} . Let now

$$b_A \doteq b \in \mathcal{B} \text{ s.t. } m_b(A) = 1, m_b(B) = 0 \forall B \neq A \quad (4)$$

be the unique b.f. assigning all the mass to a single subset A of Θ (A -th *basis* belief function). It can be proven that [8], denoting by \mathcal{E}_b the list of focal elements of b ,

Theorem 1. *The set of all the belief functions with focal elements in a given collection L is closed and convex in \mathcal{B} :*

$$\{b : \mathcal{E}_b \subseteq L\} = Cl(b_A : A \in L)$$

where Cl denotes the convex closure operator:

$$Cl(b_1, \dots, b_k) = \{b \in \mathcal{B} : b = \alpha_1 b_1 + \dots + \alpha_k b_k, \sum_i \alpha_i = 1, \alpha_i \geq 0 \forall i\}. \quad (5)$$

The following is then just a consequence of Theorem 1.

Corollary 1. *The belief space \mathcal{B} coincides with the convex closure of all the basis belief functions b_A : $\mathcal{B} = Cl(b_A, \emptyset \subsetneq A \subseteq \Theta)$.*

In convex geometry, an n -dimensional *simplex* is the convex closure of $n + 1$ (affinely independent [9]) points x_1, \dots, x_{n+1} of the Euclidean space \mathbb{R}^n , $Cl(x_1, \dots, x_{n+1})$. The *faces* of an n -dimensional simplex are all the possible simplices generated by a subset of its vertices, i.e. $Cl(x_{j_1}, \dots, x_{j_k})$ with $\{j_1, \dots, j_k\} \subset \{1, \dots, n + 1\}$. Its $n - 1$ dimensional faces are obtained by simply eliminating one vertex. Lower dimensional faces are obtained by erasing an arbitrary number of vertices.

As it is easy to see that the basis b.f.s b_A are affinely independent, Corollary

1 states that the belief space \mathcal{B} is a simplex (Figure 1). Moreover, each belief function $b \in \mathcal{B}$ can be written as a convex sum as

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A \quad (6)$$

i.e. the b.p.a. m_b is nothing but the set of simplicial coordinates of b in the simplex \mathcal{B} . Clearly, as a probability is a belief function assigning non zero

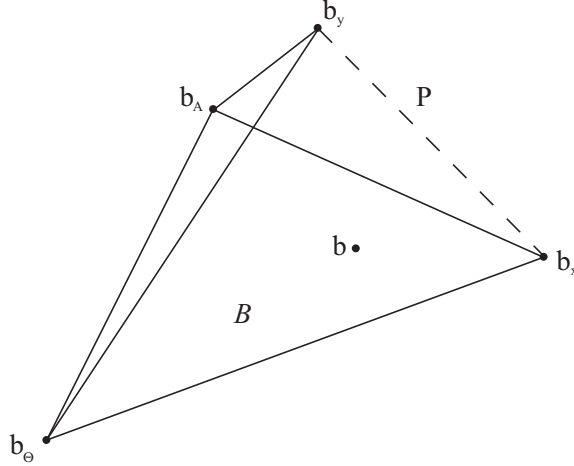


Fig. 1. Simplicial structure of the belief space \mathcal{B} : Its vertices are all the basis belief functions b_A (defined as $m_{b_A}(A) = 1$, $m_{b_A}(B) = 0 \forall B \neq A$) represented as vectors of the Cartesian space \mathbb{R}^{N-1} . The probabilistic subspace \mathcal{P} (the region of Bayesian belief functions) is just a subset $Cl(b_x, x \in \Theta)$ of its border.

masses to singletons only, Theorem 1 implies that the set \mathcal{P} of all Bayesian b.f.s is a subset of the border of \mathcal{B} , precisely the simplex determined by all basis b.f.s associated with singletons²: $\mathcal{P} = Cl(b_x, x \in \Theta)$.

Some one-dimensional faces of the belief space have also an intuitive meaning in terms of belief. Consider the segments $Cl(b_\theta, b_A)$ joining the vacuous belief function b_θ ($m_{b_\theta}(\Theta) = 1, m_{b_\theta}(B) = 0 \forall B \neq \Theta$) with the basis b.f. b_A (4). Points of $Cl(b_\theta, b_A)$ can be written as a convex combination as $b = \alpha b_A + (1 - \alpha) b_\theta$. Since convex combinations as b.p.a.s in \mathcal{B} , such a b.f. b has b.p.a.

$$m_b(A) = \alpha, \quad m_b(\Theta) = 1 - \alpha$$

i.e. b is a *simple support function* focused on A [1].

Accordingly, the union of these segments for all events A : $\mathcal{S} = \bigcup_{A \subseteq \Theta} Cl(b_\theta, b_A)$, is the region of simple support belief functions on Θ .

² With a harmless abuse of notation we denote the basis b.f. associated with a singleton x by b_x instead of $b_{\{x\}}$, and write $m_b(x), pl_b(x)$ instead of $m_b(\{x\}), pl_b(\{x\})$.

2.2.1 Binary frame

As an example let us consider a frame of discernment containing only two elements, $\Theta_2 = \{x, y\}$. In this very simple case each b.f. $b : 2^{\Theta_2} \rightarrow [0, 1]$ is completely determined by its belief values $b(x), b(y)$, as $b(\Theta) = 1$ and $b(\emptyset) = 0 \forall b$. We can then collect them in a vector of $\mathbb{R}^{N-2} = \mathbb{R}^2$ (since $N = 2^2 = 4$):

$$[b(x) = m_b(x), b(y) = m_b(y)]' \in \mathbb{R}^2. \quad (7)$$

Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ we can easily infer that the set \mathcal{B}_2 of all the possible belief functions on Θ_2 can be depicted as the triangle in the Cartesian plane of Figure 2, whose vertices are the points

$$b_\Theta = [0, 0]', b_x = [1, 0]', b_y = [0, 1]'$$

which correspond (through Equation (7)) respectively to the vacuous belief function b_Θ ($m_{b_\Theta}(\Theta) = 1$), the Bayesian b.f. b_x with $m_{b_x}(x) = 1$, and the Bayesian b.f. b_y with $m_{b_y}(y) = 1$. The region \mathcal{P}_2 of all Bayesian b.f.s on Θ_2

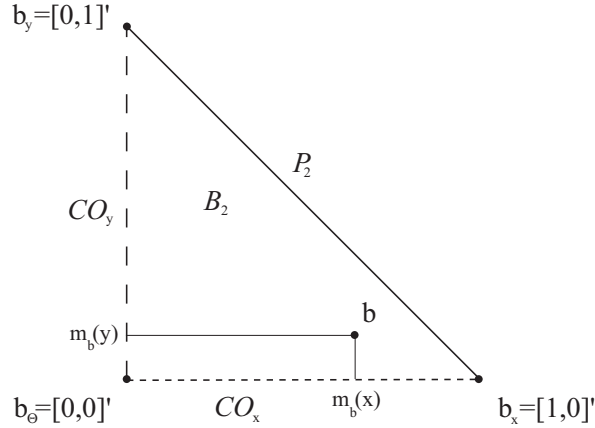


Fig. 2. The belief space \mathcal{B} for a binary frame is a triangle in \mathbb{R}^2 whose vertices are the basis belief functions focused on $\{x\}, \{y\}$ and Θ , (b_x, b_y, b_Θ) respectively. The probability region is the segment $Cl(b_x, b_y)$, while consonant and consistent belief functions are constrained to belong to the union of the two segments $\mathcal{CS}_x = \mathcal{CO}_x = Cl(b_\Theta, b_x)$ and $\mathcal{CS}_y = \mathcal{CO}_y = Cl(b_\Theta, b_y)$.

is in this case the diagonal line segment $Cl(b_x, b_y)$. On the other side, simple support functions focused on $\{x\}$ lie on the horizontal segment $Cl(b_\Theta, b_x)$, while simple support b.f. focused on $\{y\}$ form the vertical segment $Cl(b_\Theta, b_y)$.

On $\Theta_2 = \{x, y\}$ consonant belief functions can have as chain of focal elements one between $\{\{x\}, \Theta_2\}$ and $\{\{y\}, \Theta_2\}$. As a consequence, all co.b.f.s on Θ_2 are simple support functions, and their region \mathcal{CO}_2 is the union of two segments

$$\mathcal{CO}_2 = \mathcal{S}_2 = \mathcal{CO}_x \cup \mathcal{CO}_y = Cl(b_\Theta, b_x) \cup Cl(b_\Theta, b_y).$$

In the rest of the paper we will study the geometry of consonant belief functions in the general case of arbitrary frames, and relate their behavior to that of Bayesian b.f.s, delineating some sort of *duality* between probabilities and possibilities. As we will see in the following, this peculiar relation manifests itself in two ways: a dual geometry based on the notion of *simplicial complex*, and the intrinsic link between Bayesian and consonant functions with two different norms in the belief space.

2.3 The consonant complex

Using the language of convex geometry we can pose the problem of finding the region of \mathcal{B} whose points correspond to consonant belief functions. It turns out that this subspace can be described as a *simplicial complex*, a notion dual to that of simplex. We first need to focus on the behavior of *chains* of subsets in the belief space.

Let us consider, as an example, the case of a frame of size 3: $\Theta = \{x, y, z\}$. Belief functions $b \in \mathcal{B}_3$ can be written as 6-dimensional vectors

$$[b(x), b(y), b(z), b(\{x, y\}), b(\{x, z\}), b(\{y, z\})]'$$

after neglecting the constant coordinate $b(\Theta_3) = 1 \forall b \in \mathcal{B}_3$ associated with $B = \Theta$. The maximal chains are then

$$\begin{array}{lll} \{x\} \subset \{x, z\} \subset \Theta & \{y\} \subset \{x, y\} \subset \Theta & \{z\} \subset \{y, z\} \subset \Theta \\ \{x\} \subset \{x, y\} \subset \Theta & \{y\} \subset \{y, z\} \subset \Theta & \{z\} \subset \{x, z\} \subset \Theta \end{array}$$

Each singleton is then associated with 2 chains, and the total number of convex components, whose dimension is $|\Theta| - 1 = 2$, is $3! = 6$:

$$\begin{array}{lll} Cl(b_x, b_{\{x,z\}}, b_\Theta) & Cl(b_y, b_{\{x,y\}}, b_\Theta) & Cl(b_z, b_{\{y,z\}}, b_\Theta) \\ Cl(b_x, b_{\{x,y\}}, b_\Theta) & Cl(b_y, b_{\{y,z\}}, b_\Theta) & Cl(b_z, b_{\{x,z\}}, b_\Theta) \end{array}$$

The reader can realize how each 2-dimensional simplex (for instance $Cl(b_x, b_{\{x,z\}}, b_\Theta)$) has an intersection of dimension $|\Theta| - 2 = 1$ ($Cl(b_{\{x,z\}}, b_\Theta)$) with a single other component ($Cl(b_z, b_{\{x,z\}}, b_\Theta)$) associated with a different element of Θ .

The geometry of the ternary frame can then be represented as in Figure 3, where the belief space is 6-dimensional $\mathcal{B}_3 = Cl(b_x, b_y, b_z, b_{\{x,y\}}, b_{\{x,z\}}, b_{\{y,z\}}, b_\Theta)$, its probabilistic subspace is a 2-dimensional simplex $\mathcal{P}_3 = Cl(b_x, b_y, b_z)$, and the consonant subspace \mathcal{CO}_3 is given by the union of the connected components listed above. These properties of \mathcal{CO} can be summarized by means of another concept of convex geometry, immediately derived from that of simplex [9].

Fig. 3. The simplicial complex \mathcal{CO}_3 of the consonant belief functions for a ternary frame Θ_3 . The complex is composed by $n! = 3! = 6$ convex components of dimension $n - 1 = 2$, each vertex of \mathcal{P}_3 being shared by $(n - 1)! = 2! = 2$ of them. The region is connected, and is part of the border $\partial\mathcal{B}_3$ of the belief space \mathcal{B}_3 .

Definition 5. A simplicial complex is a collection Σ of simplices of arbitrary dimensions possessing the following properties:

- (1) if a simplex belongs to Σ , then all its faces of any dimension belong to Σ ;
- (2) the intersection of two simplices is a face of both.

Let us consider for instance two triangles on the plane (2-dimensional simplices). Roughly speaking, the second condition says that the intersection of those triangles cannot contain points of their interiors (Figure 4 left). It can-

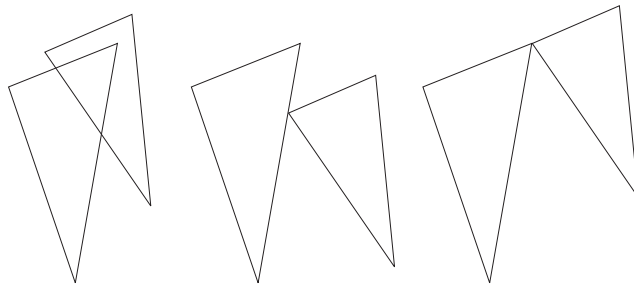


Fig. 4. Constraints on the intersection of simplices in a complex. Only the right-hand pair meets condition (2) of the definition of simplicial complex.

not also be any subset of their borders (middle), but has to be a face (right, in this case a single vertex). Note that if two simplices intersect in a face τ , they obviously intersect in every face of τ .

Proposition 2. \mathcal{CO} is a simplicial complex included in the belief space \mathcal{B} .

As Figure 3 shows, \mathcal{P} and the convex components of \mathcal{CO} have the same di-

mension, and are both parts of the boundary $\partial\mathcal{B}$ of the belief space.

3 Consonant approximation in the binary belief space

3.1 Duality of norms and distances

The geometric description of consonant belief functions clearly pictures a sort of duality between probability and possibility measures, represented by the dichotomy simplex - simplicial complex. It is not hard to show that this is due to the connection of those measures with the norms L_1 and L_∞ respectively, i.e. $P(A) = \sum_{x \in A} P(x)$, $Pos(A) = \max_{x \in A} Pos(x)$. Interesting consequences of this duality can be appreciated in the approximation problem.

3.2 The approximation problem

Consider as usual the binary case (Figure 5), and find the probabilistic and consonant approximations of a b.f. $b \in \mathcal{B}_2$, using the classical norms L_1, L_2, L_∞ :

$$\begin{aligned} \|b - b'\|_{L_1} &= \sum_{A \subseteq \Theta} |b(A) - b'(A)|, & \|b - b'\|_{L_2} &= \sqrt{\sum_{A \subseteq \Theta} (b(A) - b'(A))^2}, \\ \|b - b'\|_{L_\infty} &= \max_{A \subseteq \Theta} \{|b(A) - b'(A)|\} \end{aligned}$$

In the Bayesian case

$$p_{L_2}[b] \doteq \arg \min_{p \in \mathcal{P}} \|b - p\|_{L_2} = \left[m_b(x) + \frac{m_b(\Theta)}{2}, m_b(y) + \frac{m_b(\Theta)}{2} \right]';$$

this probability is called *orthogonal projection* $\pi[b]$ of b onto \mathcal{P} [10], and coincides with the pignistic function $BetP[b]$ [11–13] in the binary case. The L_∞ norm yields as approximation the same Bayesian b.f.

$$\begin{aligned} p_{L_\infty}[b] &\doteq \arg \min_{p \in \mathcal{P}} \|b - p\|_{L_\infty} = \arg \min_{p \in \mathcal{P}} \max\{|b(x) - p(x)|, |b(y) - p(y)|\} = \\ &= \arg \min_{p \in \mathcal{P}} \max\{|m_b(x) - p(x)|, |m_b(y) - p(y)|\} = \\ &= \left[m_b(x) + \frac{m_b(\Theta)}{2}, m_b(y) + \frac{m_b(\Theta)}{2} \right]' = p_{L_2}[b] \end{aligned}$$

while the optimization problem

$$\begin{aligned} \arg \min_{p \in \mathcal{P}} \|b - p\|_{L_1} &= \arg \min_{p \in \mathcal{P}} (|b(x) - p(x)| + |b(y) - p(y)|) = \\ &= \arg \min_{p \in \mathcal{P}} (|m_b(x) - p(x)| + |m_b(y) - p(y)|) \end{aligned}$$

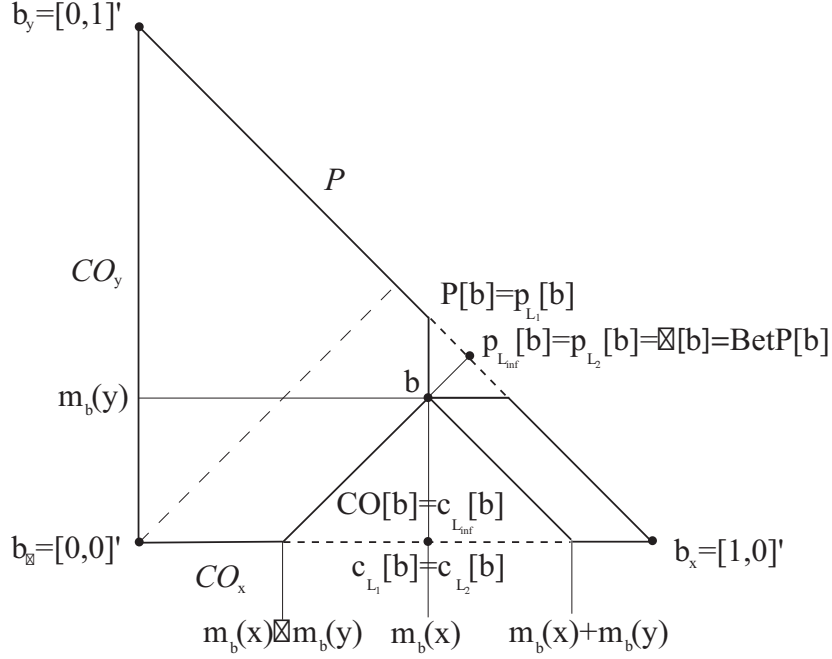


Fig. 5. The dual behavior of probabilities and possibilities in the approximation problem. Bayesian ($p_{L_i}[b]$) and consonant ($c_{L_i}[b]$) approximations of a b.f. b associated with the norms L_1, L_2, L_∞ are shown.

has as solution the whole set of probabilities *compatible* with b [14,15], i.e.

$$\{p \in \mathcal{P} : p(A) \geq b(A) \forall A \subseteq \Theta\}. \quad (8)$$

The corresponding minimum is

$$\begin{aligned} |m_b(x) - p(x)| + |m_b(y) - p(y)| &= p(x) - m_b(x) + p(y) - m_b(y) = \\ &= p(x) + p(y) - m_b(x) - m_b(y) = 1 - m_b(x) - m_b(y) = m_b(\Theta). \end{aligned}$$

In the consonant case, on the other side, we need first to find an approximation on *each* convex component of the consonant subspace, and then select the optimal solution among those “partial” approximations.

3.2.1 L_1, L_2 norms

We get for L_2 :

$$c_{L_2}[b] \doteq \arg \min_{c \in \mathcal{CO}} \|b - c\|_{L_2} = \begin{cases} [m_b(x), 0]' & m_b(x) \leq m_b(y) \\ [0, m_b(y)]' & m_b(x) \geq m_b(y) \end{cases} \quad (9)$$

(see Figure 5), while

$$\|b - c\|_{L_1} = |m_b(x) - c(x)| + |m_b(y) - c(y)| = |m_b(x) - c(x)| + m_b(y)$$

for $c \in \mathcal{CO}_x$, which is minimum when $c(x) = m_b(x)$ ($c(y) = 0$ by definition). Analogously for the component \mathcal{CO}_y

$$\arg \min_{c \in \mathcal{CO}_y} \|b - c\|_{L_1} = [0 \ m_b(y)]'$$

so that $c_{L_1}[b] = c_{L_2}[b]$ is again given by Equation (9).

3.2.2 L_∞ norm

The case of the L_∞ norm is more intriguing. For $c \in \mathcal{CO}_x$ the L_∞ distance between b and c is

$$\|b - c\|_{L_\infty} = \max\{|m_b(x) - c(x)|, |m_b(y) - c(y)|\} = \max\{|m_b(x) - c(x)|, m_b(y)\}$$

whose minimum corresponds to all consonant belief functions such that $|m_b(x) - c(x)| \leq m_b(y)$, i.e.

$$\arg \min_{c \in \mathcal{CO}_x} \|b - c\|_{L_\infty} = \{c \in \mathcal{CO}_x : \max\{0, m_b(x) - m_b(y)\} \leq c(x) \leq m_b(x) + m_b(y)\}.$$

In the same manner,

$$\arg \min_{c \in \mathcal{CO}_y} \|b - c\|_{L_\infty} = \{c \in \mathcal{CO}_y : \max\{0, m_b(y) - m_b(x)\} \leq c(y) \leq m_b(y) + m_b(x)\}$$

so that we can write

$$\begin{aligned} \arg \min_{c \in \mathcal{CO}_2} \|b - c\|_{L_\infty} &= CO[b] \doteq \\ &\doteq \begin{cases} \{c \in \mathcal{CO}_x : m_b(x) - m_b(y) \leq c(x) \leq m_b(x) + m_b(y)\}, & m_b(x) \geq m_b(y) \\ \{c \in \mathcal{CO}_y : m_b(y) - m_b(x) \leq c(y) \leq m_b(y) + m_b(x)\} & m_b(y) \geq m_b(x) \end{cases} \end{aligned}$$

since when $m_b(x) \geq m_b(y)$ $\max\{0, m_b(x) - m_b(y)\} = m_b(x) - m_b(y)$, while when $m_b(y) \geq m_b(x)$ $\max\{0, m_b(y) - m_b(x)\} = m_b(y) - m_b(x)$.

It suffices to compare the expressions of the consonant approximations $c_{L_1}[b]$, $c_{L_2}[b]$, $c_{L_\infty}[b]$ of b to note that $c_{L_1}[b] = c_{L_2}[b]$ is the *center of mass* of the above set $CO[b]$ (see Figure 5 again):

$$c_{L_1}[b] = c_{L_2}[b] = \overline{CO}[b].$$

3.2.3 Duality

Summarizing the results we obtained for both Bayesian and consonant approximations

$$\begin{aligned} c_{L_\infty}[b] &= CO[b] & p_{L_1}[b] &= P[b] \\ c_{L_2}[b] &= \overline{CO}[b] & p_{L_2}[b] &= \overline{P}[b] \\ c_{L_1}[b] &= \overline{CO}[b] & p_{L_\infty}[b] &= \overline{P}[b] \end{aligned} \quad (10)$$

we can easily recognize the dual role of the norms L_∞ and L_1 in the two problems. Looking at Equation (10) it is natural to call the set $CO[b]$ the collection of *consonant belief functions compatible with b*

3.3 Compatible consonant belief functions

We can try and give a characterization of compatible co.b.f.s in terms of some inclusion relation similar to that of $P[b]$ (8). Observing Figure 6 we can note

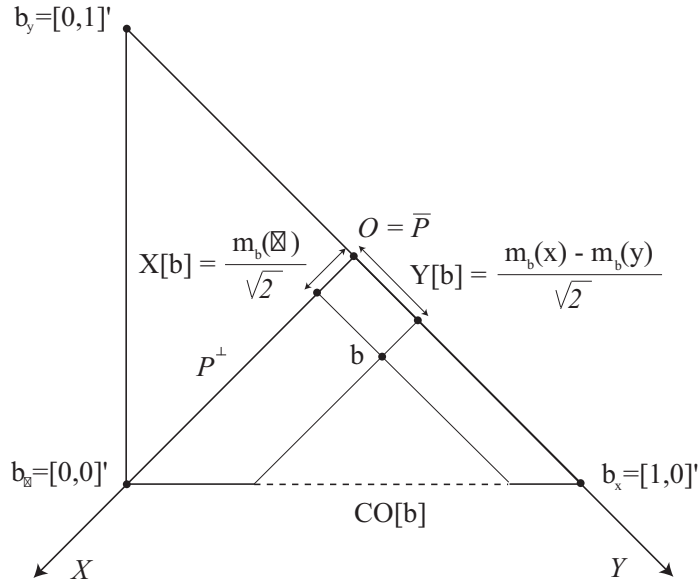


Fig. 6. Characterization of compatible consonant belief functions in terms of the reference frame (X, Y) formed by the probability line and the line \mathcal{P}^\perp orthogonal to \mathcal{P} in $\overline{\mathcal{P}} = [1/2 \ 1/2]'$.

that in the binary case $CO[b]$ is the set of co.b.f.s for which

$$X(c) \geq X(b), \quad Y(c) \geq Y(b)$$

in the reference frame (X, Y) with origin $O = \overline{\mathcal{P}} = [1/2, 1/2]'$. The coordinates of a belief function b in this reference frame can be computed through simple

trigonometric arguments, and are given by

$$X(b) = \frac{m_b(\Theta)}{\sqrt{2}} = \frac{1-m_b(x)-m_b(y)}{\sqrt{2}}, \quad Y(b) = \frac{m_b(x)-m_b(y)}{\sqrt{2}}.$$

More interestingly, it is easy to recognize from Figure 6 that $X(b)$ is nothing but the L_2 distance of b from the Bayesian region, while $Y(b)$ is the distance between b and the orthogonal complement of \mathcal{P} in $\bar{\mathcal{P}}$:

$$X(b) = \|b - \mathcal{P}\|_{L_2} \quad Y(b) = \|b - \mathcal{P}^\perp\|_{L_2}.$$

Furthermore, \mathcal{P}^\perp (or better its segment $Cl(b_\Theta, \bar{\mathcal{P}})$ joining b_Θ and $\bar{\mathcal{P}}$) is the set of belief functions in which the mass is equally distributed among events of the same size ($\{x\}$ and $\{y\}$ in the binary case). This link between orthogonality and equidistribution is true in the general case too. In [16] we proved that

Proposition 3. *If a belief function b is such that*

$$pl_b(x; k) \doteq \sum_{A \supset x, |A|=k} m_b(A) = const = pl_b(\cdot; k)$$

for all $k = 1, \dots, |\Theta| - 1$ then b is orthogonal to the probabilistic subspace, $b \perp \mathcal{P}$.

In conclusion we can claim that, at least in the binary case,

Theorem 2. *The consonant belief functions compatible with $b \in \mathcal{B}_2$ are all the co.b.f.s which are at the same time less Bayesian and less equally distributed (less orthogonal to \mathcal{P}) than b .*

It remains to decide how this result generalizes for arbitrary frames.

3.4 Geometry of inner and outer consonant approximations

It may be interesting to compare the above results with the classical inner and outer consonant approximations [17], at least in this simple case study.

3.4.1 Inner approximations

Inner consonant approximations of a b.f. b are those co.b.f.s such that $c(A) \geq b(A) \forall A \subseteq \Theta$ (or equivalently $pl_c(A) \leq pl_b(A) \forall A$). Such an approximation exists iff b is consistent, i.e. the intersection of all its focal elements is non-empty. In the binary case this means that inner approximations of b exist iff b is already consonant: $b \in \mathcal{CO}_x$ or $b \in \mathcal{CO}_y$.

The optimal inner approximation is the co.b.f. \hat{c} such that $pl_{\hat{c}}(x) = pl_b(x) \forall x \in \Theta$. However, when b is *not* consonant a “focused consistent transforma-

tion” can be applied to get a new belief function b' such that

$$m'(A \cup x_i) = m(A) \quad \forall A \subseteq \Theta$$

and x_i is the element of Θ with highest plausibility.

In the binary example this yields simply the orthogonal projection $c_{L_2}[b]$ of b onto \mathcal{CO} . Suppose in fact $pl_b(x) \geq pl_b(y)$ ($m_b(x) \geq m_b(y)$). Then m' is such that

$$m'(x) = m_b(x), \quad m'(\Theta) = m'(\{y\} \cup \{x\}) + m'(\Theta \cup \{x\}) = m_b(y) + m_b(\Theta).$$

The corresponding belief function b' is depicted in Figure 7.

3.4.2 Outer approximations

Outer consonant approximations of a b.f. b are those co.b.f.s such that $c(A) \leq b(A) \quad \forall A \subseteq \Theta$ (or equivalently $pl_c(A) \geq pl_b(A) \quad \forall A$). The set $O[b]$ of the outer consonant approximations of b in the binary belief space \mathcal{B}_2 is also depicted in Figure 7 (red region). Optimal outer approximations (maximal with respect

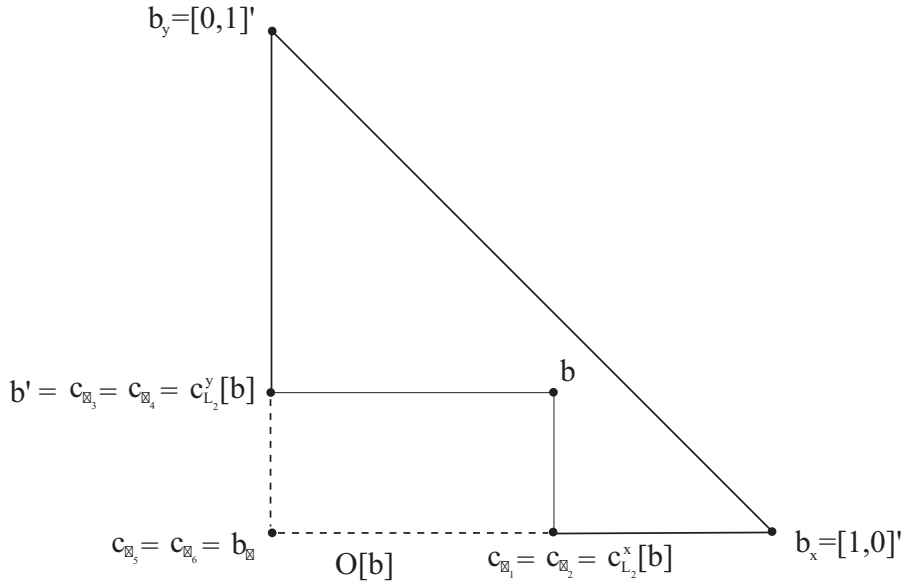


Fig. 7. Geometry of outer consonant approximations of a belief function $b \in \mathcal{B}_2$. The maximal outer approximations are nothing but the orthogonal projections of b onto the two convex components $\mathcal{CO}_x, \mathcal{CO}_y$ of the consonant subspace \mathcal{CO} .

to the weak inclusion relation (3)) can be found by applying an iterative procedure in which all permutations ρ of the focal elements $\{E_1, \dots, E_k\}$ of b are considered

$$\{E_{\rho(1)}, \dots, E_{\rho(k)}\}$$

and the following family of sets is introduced

$$\{S_1^\rho = E_{\rho(1)}, S_2^\rho = E_{\rho(1)} \cup E_{\rho(2)}, \dots, S_k^\rho = E_{\rho(1)} \cup \dots \cup E_{\rho(k)}\}$$

so that a new belief function c_ρ can be defined with b.p.a.

$$m_{c_\rho}(S_j^\rho) = \sum_{i: \min\{l: E_i \subseteq S_j^\rho\} = j} m_b(E_i).$$

Without getting too much into the details, the possible permutations of the focal elements $\{\{x\}, \{y\}, \Theta\}$ of b in the binary case are six: It is easy to check that they correspond to the outer approximations $c_{\rho_1}, \dots, c_{\rho_6}$ in Figure 7, i.e. the orthogonal projections $c_{L_2}^x[b], c_{L_2}^y[b]$ of b onto $\mathcal{CO}_x, \mathcal{CO}_y$ respectively, and the vacuous b.f. $b_\Theta = \mathbf{0}$. Optimal outer approximations are then $c_{L_2}^x[b]$ and $c_{L_2}^y[b]$.

3.5 Some conjectures on the L_i consonant approximation problem

The stimulating discussion of the case of binary frames leaves a series of open issues. One of course is the characterization of compatible c.b.f in the general case, while general expressions for all consonant approximations based on the norms L_1, L_2, L_∞ need to be found. The problem is complicated (with respect to the Bayesian case) by the necessity of checking the distance between the original b.f. b and all the convex components of the consonant subspace.

Several properties need to be tested: we want to understand

- whether all the approximations fall on the same convex component of \mathcal{CO} , as they do in the binary case;
- if it is true in general that $c_{L_1}[b] = c_{L_2}[b]$, and whether some of them coincide with the center of mass of compatible consonant belief functions.

4 Consonant approximation based on the L2 norm

4.1 Ternary case

4.2 Orthogonal projection onto a maximal simplex of CO

4.2.1 Linear system

4.2.2 Expression of \mathbf{b}

4.2.3 Structure of \mathcal{A}

4.2.4 Solution

4.2.5 Interpretation

4.3 Distance from all projections and minimum

4.4 Interpretation of the solution

5 Consonant approximation based on the L1 norm

5.1 Ternary case

5.2 Problem formulation

5.3 Solution in terms of nodes

Th pag 44 notes

5.4 Expression in terms of b.p.a.

TO DO, pages 44-46 notes

5.5 *Distance from all projections and minimum*

TO DO

5.6 *Interpretation of the solution*

TO DO

6 **Consonant approximation based on the L_∞ norm**

6.1 *Problem formulation*

pag 58 notes

6.2 *Solution*

6.3 *Interpretation*

7 **Comments and comparison**

do they all fall on the same component

are they geometrically related to each other?

8 **Conclusions**

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