International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems

GEOMETRY AND COMBINATORICS OF PLAUSIBILITY AND COMMONALITY FUNCTIONS

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In this work we extend the geometric approach to the theory of evidence in order to study the geometric behavior of the two quantities inherently associated with a belief function. i.e. the plausibility and commonality functions. After introducing the analogous of the basic probability assignment for plausibilities and commonalities, we exploit it to understand the simplicial form of both plausibility and commonality spaces. Given the intuition provided by the binary case we prove the congruence of belief, plausibility, and commonality spaces for both standard and unnormalized belief functions, and describe the point-wise geometry of these sum functions in terms of the rigid transformation mapping them onto each other. This leads us to conjecture that the D-S formalism may be in fact a geometric calculus in the line of geometric probability, and opens the way to a wider application of discrete mathematics to subjective probability.

Keywords: Theory of evidence; convex geometry; Moebius inverse; basic plausibility and commonality assignment; plausibility and commonality spaces; congruence.

1. Introduction

Uncertainty measures have a major role in fields like artificial intelligence and computer vision, where problems requiring formalized reasoning are common. Their applications include sensor fusion [31], target tracking [9], robotics and autonomous navigation [47], data mining [45], economics [32], and pattern recognition [2,8,60,28].
During the last decades a number of different descriptions of uncertain states of knowledge have been proposed, as either alternatives to or extensions of classical probability theory. They range from probability intervals [38] to credal sets [66], to monotone capacities [64], to random sets [33,48]. New original foundations of subjective probability in behavioral terms [63,62] or by means of game theory [55] have been proposed.

The *theory of evidence* (ToE) is one the most popular formalisms, thanks perhaps to the fact that it is a quite natural extension of the classical Bayesian formalism. It was introduced in the late Seventies by Glenn Shafer [52] as a way of representing epistemic knowledge, starting from a sequence of seminal works [23,24,25] of Arthur Dempster. In this formalism the best representation of chance is a *belief*

function (b.f.) rather than a Bayesian mass distribution, assigning probability values to *sets* of possibilities rather than single events.
Variants or continuous extensions of the ToE in terms of hints [43] or allocations of probability [53] have also been proposed, while the relationship between belief and probability is the focus of Smets' "transferable belief model" [58].

We recently developed a geometric approach to the theory of evidence in which belief functions are represented by points of a convex space called *belief space* [22,20]. As a matter of fact, as a belief function $b : 2^\Theta \rightarrow [0,1]$ is completely specified by its $N-1$ belief values

$$\{b(A), A \subseteq \Theta, A \neq \emptyset\},$$

$N \doteq 2^{|\Theta|}$, it can be represented as a point of $\mathbb{R}^{N-1}$.
In this framework problems concerning inference, approximation, and coherence can be posed. For instance, the geometry of Dempster's rule of combination can be described in the belief space as an intersection of linear subspaces [17]: A natural and promising prosecution of this line of research is the analysis of other operators like the disjoint rule of combination, or natural extension [62].
The problem of approximating a given belief function with a probability [4,61,56] or a possibility [29] can also be studied in a geometric context. We recently conducted a comparative study [16] of several Bayesian approximations of belief functions, introducing two new Bayesian b.f. (the *orthogonal projection* of b onto $\mathcal{P}$, and the *intersection probability*) from purely geometrical arguments. We formulated an approximation criterion [14] based on the rule of combination, whose solution we conjectured would be another well-known approximation, the relative plausibility of singletons.
The convex geometry of possibility measures (or their counterparts in the theory of evidence, consonant [21] and consistent [18] belief functions) turns out to be expressed in terms of simplicial complexes, delineating intriguing links with the notion of independence and matroid theory.

The study of the geometry of fuzzy measures, besides, has been recently faced by other researchers [36,7,6]. P. Black, in particular, has dedicated its doctoral thesis to the study of the geometry of belief functions and other monotone capacities [6]. Another close reference is perhaps a work of Ha and Haddawy [36] where methods of convex geometry are exploited to represent probability intervals in a computationally efficient fashion, by means of a data structure called *pcc-tree*. On their side Melkonyan et al. [46] use results from convex geometry to obtain representations of the prior and posterior degrees of imprecision in terms of width functions and difference bodies. These convex representations are then used to build algorithms to compute prior and posterior degrees of imprecision. In robust Bayesian statistics, in general, there is a large literature on the study of convex sets of probability distributions [13,5,39,51].

Even though the geometric approach was originally motivated by the approximation problem, it now clear that it is just a symptom of a strict relationship between

combinatorics and subjective probability. Subjective probability and combinatorics are apparently unrelated fields. However belief functions, as they are functions defined on power sets, are inherently related to a number of topics of combinatorics like Boolean algebras, partially ordered sets and lattices[35], matroids[19] just to cite a few of them. These links have never been systematically explored, even though some work has been recently done in this direction [35,34,41,18], specially by M. Grabish and his group.
In this perspective we go in this work a step further and extend the geometric approach to the theory of evidence in order to study the geometric behavior of the two quantities which carry the same evidence associated with a belief function: plausibility (pl.f.) and commonality (comm.f.) functions. We show that they share the same combinatorial structure of *sum function*[1] and introduce their Moebius inverses called basic plausibility and commonality assignments. This will allows us to prove their simplicial geometry and realize that the simplices they form are congruent, so that the probabilistic relation between upper and lower probabilities can be geometrically expressed as a rigid transformation.
This eventually places a new element in the geometric semantics of the theory of evidence. As belief functions are points of a simplex[22], possibility measures form a simplicial complex[21], and Dempster's rule itself is nothing but an intersection of linear spaces[17] it may well be that the Dempster-Shafer formalism is in fact some form of geometric calculus.

It is possible that this could eventually lead us to a confluence with the field of geometric probability or continuous combinatorics[42], which studies invariant measures on sets of geometric objects (which generalize the concept of volume) and relates them to additive probability measures. As an example of possible cross-fertilization coming from an interdisciplinar approach we can cite the well-known result stating that all simplicial complexes on a partially ordered set form a distributive lattice[42]. As belief, plausibility and commonality functions, but also consonant and consistent b.f. (finite possibility measures) form simplicial complexes in 2^Θ this could bring to an algebraic interpretation of the mutual relations between all those measures which mirrors the geometric one.

1.1. *Paper outline*

First (Section 2) we recall the basic notions of the theory of evidence, in particular the key ideas of belief and plausibility functions (together with that of commonality function), which can be interpreted as lower and upper limits to the probability values of a convex set of probability measures. In Section 3.1 we present the geometric approach to the ToE and the concept of belief space $\mathcal{B}$, as the set of all the belief functions in the appropriate Cartesian space, and extend the idea of belief space to the case of unnormalized belief functions (u.b.f., Section 3.4), i.e. b.f. which assign non-zero mass to the empty set.
Then, analogously to what done for belief functions, we introduce the notions of

basic plausibility (Section 4) and commonality (5) assignments as Moebius inverses of pl.f. and comm.f. respectively, proving by doing so that both plausibility and commonality functions are in fact sum functions on the partially ordered set 2^Θ. This allows us to extend the geometric description to plausibility and commonality functions, and recover the convex structure of plausibility $\mathcal{PL}$ and commonality $\mathcal{Q}$ spaces, whose vertices are given a straightforward interpretation in terms of plausibilities and commonalities associated with basis belief functions.
Given the intuition provided by the case of binary frames we then pass to analyze the relationships between those simplices (Section 6), by proving in particular the congruence of $\mathcal{B}$ and $\mathcal{PL}$ (in the case of both standard and unnormalized belief functions, 6.1) and of the pair $\mathcal{PL}^U$, $\mathcal{Q}^U$ (for u.b.f. only, 6.2). Finally (7), we discuss the point-wise geometry of the triplet (b, pl_b, Q_b) in terms of the rigid transformation mapping them onto each other, as the geometric counterpart of the duality between upper and lower probabilities in subjective probability.

A running example concerning the simple case of binary frames is used throughout the paper to illustrate and anticipate the formal analysis of the properties of pl.f. and comm.f. Besides, some of the proofs are moved to an Appendix to improve the readability of the paper.

2. The theory of evidence

The *theory of evidence* [52] has been introduced in the late Seventies by Glenn Shafer as a way of representing epistemic knowledge, starting from a sequence of seminal works [23,24,25], of Arthur Dempster.

2.1. *Belief functions as sum functions, basic probability assignments*

Even though belief functions can be given several alternative but equivalent definitions in terms of multi-valued mappings[54], random sets[48,49,40], inner measures[50,30], in Shafer's formulation[52] a central role is played by the notion of *basic probability assignment.*

Definition 1. A *basic probability assignment* (b.p.a.) over a finite set (*frame of discernment* [52]) Θ is a function $m : 2^\Theta \to [0,1]$ on its power set $2^\Theta = \{A \subset \Theta\}$ such that

$$m(\emptyset) = 0, \ \sum_{A \subseteq \Theta} m(A) = 1, \ \ m(A) \geq 0 \ \ \forall A \subset \Theta.$$

Subsets of Θ associated with non-zero values of m are called *focal elements.*

Definition 2. The *belief function* $b : 2^\Theta \to [0,1]$ associated with a basic probability assignment m on Θ is defined as:

$$b(A) = \sum_{B \subseteq A} m(B). \tag{1}$$

In combinatorics, functions of the form (1) on a partially ordered set are called *sum functions* [1]. A belief function b is then the sum function associated with a basic probability assignment m_b on the partially ordered set $(2^\Theta, \subseteq)$.
Conversely, the unique basic probability assignment m_b associated with a given belief function b can be recovered by means of the *Moebius inversion formula*

$$m_b(A) = \sum_{B \subseteq A} (-1)^{|A-B|} b(B) \tag{2}$$

so that there is a 1-1 correspondence between the two set functions $m_b \leftrightarrow b$.
A sum function can be seen as the discrete counterpart of the indefinite integral in calculus, so that we may view Moebius inversion as the discrete counterpart of the derivative.

In the theory of evidence a probability function is simply a special belief function assigning non-zero masses only to singletons (*Bayesian* b.f.): $m_b(A) = 0$, $|A| > 1$.

2.2. *Plausibility and commonality functions*

A dual mathematical representation of the evidence encoded by a belief function b is the *plausibility function* (pl.f.)

$$\begin{array}{rcl} pl_b : 2^\Theta & \to & [0,1] \\ A & \mapsto & pl_b(A) \end{array}$$

where the plausibility $pl_b(A)$ of an event A is given by

$$pl_b(A) \doteq 1 - b(A^c) = 1 - \sum_{B \subseteq A^c} m_b(B). \tag{3}$$

For each event A, $pl_b(A)$ expresses the amount of evidence *not against* A. pl_b conveys as much information as b, and can be computed from the b.p.a. as

$$pl_b(A) = \sum_{B \cap A \neq \emptyset} m_b(B) \geq b(A).$$

We will denote with m_b, pl_b the b.p.a. and pl.f. uniquely associated with a belief function b.

It is well known that belief functions can be interpreted as lower bounds of probabilities, i.e. each belief function b determines a class of Bayesian b.f. such that

$$p(A) \geq b(A) \; \forall A \subset \Theta.$$

It can be proven that plausibility functions are then nothing but *upper bounds* in the same convex set for the probability values of the events,

$$p(A) \leq pl_b(A) \; \forall A \subset \Theta.$$

Alternatively, the evidence carried by a belief function can be also described by the *commonality function* (comm.f.)

$$\begin{array}{rcl} Q_b : 2^\Theta & \to & [0,1] \\ A & \mapsto & Q_b(A) \end{array}$$

where the *commonality number* $Q_b(A)$ can be interpreted as the amount of mass which can move freely through the entire event A,

$$Q_b(A) \doteq \sum_{B \supseteq A} m_b(B). \tag{4}$$

2.3. *Example*

Let us consider as an example a belief function b on a frame of size 3, $\Theta = \{x, y, z\}$ with basic probability assignment (see Figure 1)

$$m_b(x) = 1/3, \;\; m_b(\Theta) = 2/3.$$

The belief values of b on all possible events of Θ are the, according to Equation

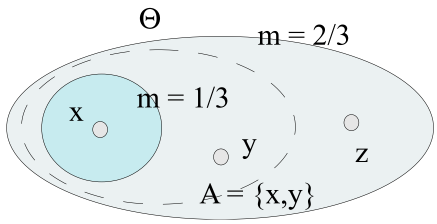

Fig. 1. The belief function of the example 2.3 has two focal elements, $\{x\}$ and Θ.

(1),

$$\begin{array}{ccc} b(x) = m_b(x) = 1/3, & b(y) = 0, & b(z) = 0, \\ b(\{x, y\}) = m_b(x) = 1/3, & b(\{x, z\}) = m_b(x) = 1/3, & b(\{y, z\}) = 0, \end{array}$$
$$b(\Theta) = m_b(x) + m_b(\Theta) = 1.$$

To appreciate the difference between belief, plausibility, and commonality values let us consider in particular the event $A = \{x, y\}$. Its belief value

$$b(\{x, y\}) = \sum_{A \subseteq \{x,y\}} m_b(A) = m_b(x) = 1/3$$

represent the amount of evidence which *surely support* $\{x, y\}$. On the other side, its plausibility value

$$pl_b(\{x, y\}) = 1 - b(\{x, y\}^c) = 1 - b(z) = 1$$

measures the evidence *not surely against* it.
Finally, the commonality number

$$Q_b(\{x, y\}) = \sum_{A \supseteq \{x,y\}} m_b(A) = m_b(\Theta) = 2/3$$

tells us which is the amount of evidence which can (possibly) *equally support* each of the outcomes in $\{x, y\}$ (i.e. x and y).

3. Geometric approach

In the theory of evidence the question of how to approximate a belief function with a probability or Bayesian b.f. naturally arises, specially in contexts in which point-wise estimates of a quantity of interest are needed. This problem has been in fact studied by many people. Several papers [65,26,27,37,59,44] have been published on this issue (see [4,3] for a review), mainly in order to find efficient implementations of the rule of combination aiming to reduce the number of focal elements. The connection between belief functions and probabilities is as well the basement of a popular approach to the theory of evidence, Smets' *pignistic* model [56], in which beliefs are represented at credal level (as convex sets of probabilities), while decisions are made by resorting to a Bayesian belief function called *pignistic transformation*. Other solutions like the *relative plausibility* function [61] have been proposed and studied [12,10,11].

However, the approximation problem can be posed in a different setting by investigating the shape of the space belief functions live, asking ourselves where do probability functions live in this space, and which is the correct distance to use to evaluate the difference between a belief function and a probability. We then introduced the notion of *belief space* ([22], [15], [20]), as the space of all the belief functions we can define on a given domain[1].

The results and scope of the geometric approach, though, go beyond the original application to the approximation problem. Even though belief functions, as they are defined on power sets, are inherently related to Boolean algebras, lattice theory, and combinatorics in general, these links have never been systematically explored. The study of the space of belief functions, as it unveils at least some of those connections to convex geometry and the theory of simplicial complexes, forces us to reflect on what this implies in terms of the deep meaning of the evidential machinery itself.

As b.f. are points of a simplex[22], possibility measures form a simplicial complex[21], and Dempster's rule itself is nothing but an intersection of linear spaces[17] it may well be that the Dempster-Shafer formalism is nothing but some cryptomorphic geometric calculus.

But let us first review the basic ideas of the geometric approach.

3.1. *The space of belief functions*

Given a frame of discernment Θ, a belief function $b : 2^\Theta \to [0,1]$ is completely specified by its $N-1$ belief values

$$\{b(A), A \subseteq \Theta, A \neq \emptyset\},$$

[1]Several notations in this paper have been changed with respect to earlier works, in order to adopt a more standard symbology for belief and plausibility functions.

$N \doteq 2^{|\Theta|}$, and can then be represented as a point of $\mathbb{R}^{N-1}$. We can introduce an orthonormal reference frame $\{X_A : A \subseteq \Theta, A \neq \emptyset\}$ so that each vector $v = \sum_{A\subseteq\Theta,A\neq\emptyset} v_A X_A$ in $\mathbb{R}^{N-1}$ is potentially a belief function, in which each component v_A measures the belief value of A: $v_A = b(A)$. However, not every point in $\mathbb{R}^{N-1}$ represents a valid b.f.

Definition 3. The *belief space* associated with Θ is the set of points $\mathcal{B}_\Theta$ of $\mathbb{R}^{N-1}$ which correspond to a belief function.

In the following we will assume the domain Θ fixed, and denote the belief space by $\mathcal{B}$. To determine which points of $\mathbb{R}^{N-1}$ "are" belief functions we can exploit the Moebius inversion formula (2), by computing the corresponding b.p.a. and checking the axioms m_b must obey. It is not difficult to prove (see [17] for the details) that $\mathcal{B}$ is convex. Let us call

$$b_A \doteq b \in \mathcal{B} \; s.t. \; m_b(A) = 1, \; m_b(B) = 0 \; \forall B \neq A$$

the unique belief function assigning all the mass to a single subset A of Θ (A-th *basis belief function*). It can be proved that [17], after denoting by $\mathcal{E}_b$ the list of focal elements of b,

Proposition 1. *The set of all the belief functions with focal elements in a given collection L is closed and convex in $\mathcal{B}$:*

$$\{b : \mathcal{E}_b \subseteq L\} = Cl(b_A : A \in L)$$

where Cl denotes the convex closure operator:

$$Cl(b_1, ..., b_k) = \{b \in \mathcal{B} : b = \alpha_1 b_1 + \cdots + \alpha_k b_k, \sum_i \alpha_i = 1, \; \alpha_i \geq 0 \; \forall i\}.$$

The following is then just a consequence of Proposition 1.

Corollary 1. *The belief space $\mathcal{B}$ coincides with the convex closure of all the basis belief functions b_A,*

$$\mathcal{B} = Cl(b_A, \; \emptyset \subsetneq A \subseteq \Theta). \tag{5}$$

3.2. *The belief space as a simplex*

An *affine combination* of k points $v_1, ..., v_k \in \mathbb{R}^N$ is a sum $\alpha_1 v_1 + \cdots + \alpha_k v_k$ whose coefficients sum to one: $\sum_i \alpha_i = 1$. The affine subspace generated by the points $v_1, ..., v_k \in \mathbb{R}^N$ is the set

$$\Big\{v \in \mathbb{R}^N : v = \alpha_1 v_1 + \cdots + \alpha_k v_k, \sum_i \alpha_i = 1\Big\}.$$

If $v_1, ..., v_k$ generate an affine space of dimension k they are said to be *affinely independent*.

Now in convex geometry, a k-dimensional *simplex* is the convex closure of $k+1$ affinely independent points $x_1, ..., x_{k+1}$ of the Cartesian space $\mathbb{R}^k$, $Cl(x_1, ..., x_{k+1})$. The *faces* of an k-dimensional simplex are all the possible simplices generated by a subset of its vertices, i.e. $Cl(x_{j_1}, ..., x_{j_m})$ with $\{j_1, ..., j_m\} \subset \{1, ..., k+1\}$. Its $k-1$ dimensional faces are obtained by simply eliminating one vertex. Lower dimensional faces are obtained by erasing an arbitrary number of vertices.

As it is easy to see that the vectors $\{b_A, \emptyset \subsetneq A \subseteq \Theta\}$ representing the basis b.f. are affinely independent in $\mathbb{R}^{N-1}$, it follows that $\mathcal{B}$ is a simplex in $\mathbb{R}^{N-1}$ (Figure 2). Moreover, each belief function $b \in \mathcal{B}$ can be written as a convex sum as follows:

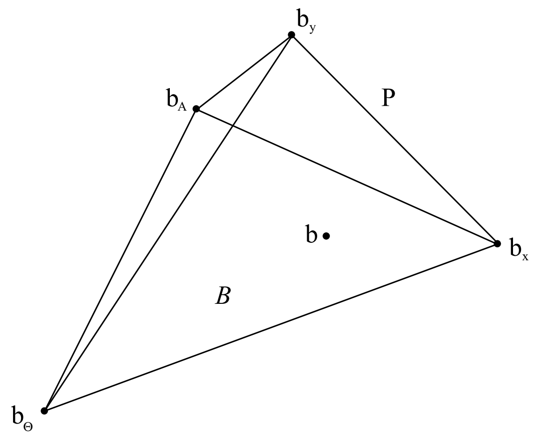

Fig. 2. Simplicial structure of the belief space $\mathcal{B}$: its vertices are all the basis belief functions b_A represented as vectors of $\mathbb{R}^N$. The probabilistic subspace is a subset $Cl(b_x, x \in \Theta)$ of its border.

$$b = \sum_{\emptyset \subsetneq A \subseteq \Theta} m_b(A) b_A. \tag{6}$$

This means that the basic probability assignment can be interpreted geometrically as the set of simplicial coordinates of b in $\mathcal{B}$. In other words, the simplicial form of $\mathcal{B}$ is the geometric counterpart of the nature of belief functions as sum functions which admit Moebius inversion.

Let us denote by $n \doteq |\Theta|$ the cardinality of Θ. Clearly, since a probability is a belief function assigning non zero masses to singletons only, Proposition 1 implies that

Corollary 2. *The set $\mathcal{P}$ of all the Bayesian belief functions is an $n-1$-dimensional face of $\mathcal{B}$, precisely the simplex determined by all the basis functions associated with singletons* [2]*:*

$$\mathcal{P} = Cl(b_x, x \in \Theta).$$

[2]With a harmless abuse of notation we denote the basis belief function associated with a

3.3. *Running example: the binary case*

To get some insight about the properties and geometric shape of the belief space it may be useful to have first a look at how belief functions defined on a frame of discernment with just two elements $\Theta_2 = \{x, y\}$ can be represented as points of a Cartesian space.

In this very simple case each belief function $b : 2^{\Theta_2} \to [0,1]$ is completely determined by its belief values $b(x), b(y)$ and $b(\Theta)$ (since $b(\emptyset) = 0$ for all b). We can then collect them in a three-dimensional vector

$$[b(x), b(y), b(\Theta)]' \in \mathbb{R}^3$$

and associate b with a point of $\mathbb{R}^3$.
However, since it is always true that $b(\Theta) = \sum_{A \subseteq \Theta} m_b(A) = 1$, the last coordinate of the vector can also be neglected (this is of course true for arbitrary frames too). In the binary case this means that we can represent b as the vector

$$[b(x) = m_b(x), b(y) = m_b(y)]' \in \mathbb{R}^2 \qquad (7)$$

of $\mathbb{R}^{N-2} = \mathbb{R}^2$ (since $N = 2^2 = 4$).
Since $m_b(x) \geq 0$, $m_b(y) \geq 0$, and $m_b(x) + m_b(y) \leq 1$ we can easily infer that the set $\mathcal{B}_2$ of all the possible belief functions on Θ_2 can be depicted as the triangle in the Cartesian plane of Figure 3, whose vertices are the points

$$b_\Theta = [0,0]', \; b_x = [1,0]', \; b_y = [0,1]'$$

which correspond (through Equation (7)) respectively to the vacuous belief function b_Θ ($m_{b_\Theta}(\Theta) = 1$), the Bayesian b.f. b_x with $m_{b_x}(x) = 1$, and the Bayesian b.f. b_y with $m_{b_y}(y) = 1$. The Bayesian belief functions on Θ_2 obey the constraint

$$m_b(x) + m_b(y) = 1$$

and can then be located as points of the segment $\mathcal{P}_2$ joining $b_x = [1,0]'$ and $b_y = [0,1]'$.

3.4. *The space of unnormalized belief functions*

In the practical use of the theory of evidence, people often consider *unnormalized belief functions* (u.b.f.) [57], i.e. belief functions admitting non-zero support $m_b(\emptyset) \neq 0$ for the empty set. The mass assigned to the empty set can be indeed interpreted as an indicator of the amount of conflict in the evidence carried by the b.f., or the possibility that the current frame of discernment does not exhaust all the possible outcomes of the problem.

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singleton x by b_x instead of $b_{\{x\}}$. Accordingly we will write $m_b(x), pl_b(x), Q_b(x)$ instead of $m_b(\{x\}), pl_b(\{x\}), Q_b(\{x\})$.

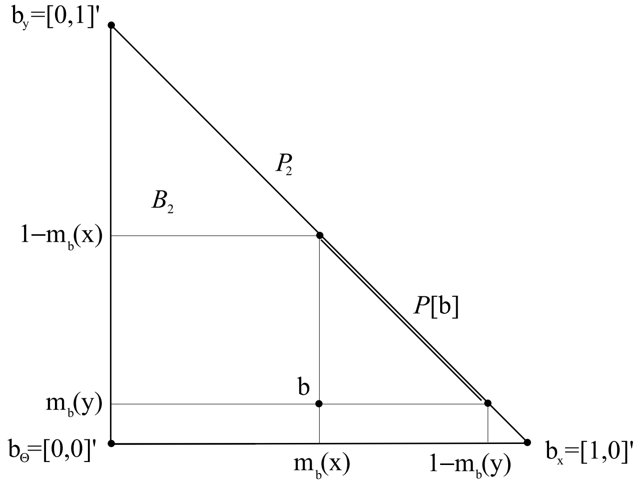

Fig. 3. The belief space $\mathcal{B}$ for a binary frame is a triangle in $\mathbb{R}^2$ whose vertices are the basis belief functions focused on $\{x\}, \{y\}$ and Θ, b_x, b_y, b_Θ respectively. The probability region is the segment $Cl(b_x, b_y)$, while the set of probabilities consistent with a b.f. b is also a segment.

Unnormalized belief functions are then naturally associated with vectors with $N = 2^{|\Theta|}$ coordinates, as $b(\emptyset)$ cannot be neglected anymore.
We can then extend the set of basis b.f. as follows

$$\{b_A \in \mathbb{R}^N, \emptyset \subseteq A \subseteq \Theta\}$$

this time including a vector $b_\emptyset \doteq [1\ 0\ \cdots 0]'$. Note also that in this case $b_\Theta = [0 \cdots 0\ 1]'$. The analysis of Section 3.1 retains though its validity: the space of unnormalized b.f. is again a simplex in $\mathbb{R}^N$, namely

$$\mathcal{B}^U = Cl(b_A, \emptyset \subseteq A \subseteq \Theta).$$

4. Convex geometry of plausibility functions

As we have seen in the first part of the paper, the simplicial structure of the space of belief functions depend on the basic probability assignment, i.e. the Moebius inverse of a belief function. The convex geometry of $\mathcal{B}$ is in other words inherently linked to the nature of b.f. as sum functions on 2^Θ.

On the other side, as plausibility and commonality functions are both equivalent representations of the evidence carried by a belief function, it is natural to guess that they should have the form of some sum function on the power set. As a matter of fact, by Moebius inversion we can define the analogous of the b.p.a. for pl.f., which is straightforward to call *basic plausibility assignment*.

4.1. *Pl.f. as sum functions: Basic plausibility assignment*

We know that some vectors of $\mathbb{R}^{N-1}$, $N \doteq |2^\Theta|$ represent actual belief functions, whose space for a simplex. We now want to understand the geometric properties of plausibility vectors $[pl_b(A), \emptyset \subsetneq A \subseteq \Theta]'$ too. A plausibility vector can indeed be expressed as

$$pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} pl_b(A) X_A \tag{8}$$

where $\{X_A : \emptyset \subsetneq A \subseteq \Theta\}$ is the orthogonal reference frame in the Cartesian space $\mathbb{R}^{N-1}$ (see also [20]).

On the other side, the basis belief functions $\{b_A : \emptyset \subsetneq A \subseteq \Theta\}$ form a set of independent vectors in $\mathbb{R}^{N-1}$, so that the collections $\{X_A\}$ and $\{b_A\}$ represent two distinct coordinate frames in the belief space. To understand the place a plausibility vector takes in the belief reference frame $\{b_A\}$ we then need to compute the coordinate change between these frames.

Theorem 1. *The coordinate change between the two coordinate frames* $\{X_A : \emptyset \subsetneq A \subseteq \Theta\}$ *and* $\{b_A : \emptyset \subsetneq A \subseteq \Theta\}$ *is given by*

$$X_A = \sum_{B \supseteq A} (-1)^{|B \setminus A|} b_B. \tag{9}$$

If we now replace expression (9) for X_A in Equation (8) we get

$$\begin{aligned} pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} pl_b(A) X_A &= \sum_{\emptyset \subsetneq A \subseteq \Theta} pl_b(A) \Big(\sum_{B \supseteq A} b_B (-1)^{|B \setminus A|} \Big) = \\ &= \sum_{\emptyset \subsetneq B \subseteq \Theta} b_B \Big(\sum_{A \subseteq B} (-1)^{|B-A|} pl_b(A) \Big) \end{aligned}$$

so that, after introducing the quantity

$$\mu_b(A) \doteq \sum_{B \subseteq A} (-1)^{|A-B|} pl_b(B) \tag{10}$$

(notice that we have inverted the role of A and B for sake of homogeneity of the notation), we can write

$$pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} \mu_b(A) b_A. \tag{11}$$

It is natural to call the function $\mu_b : 2^\Theta \to \mathbb{R}$ defined by expression (10) *basic plausibility assignment* (b.pl.a.). It is easy to recognize the Moebius equation for plausibilities: immediately

$$pl_b(A) = \sum_{B \subseteq A} \mu_b(B). \tag{12}$$

In other words plausibility functions are sum functions on 2^Θ of the form (12), whose Moebius inverse is the b.pl.a. (10). Of course, from the fact that b.f. and pl.f.

are lower and upper bounds associated with a convex set of probabilities it follows that basic probabilities and plausibilities are also related.
Namely,

Theorem 2.

$$\mu_b(A) = \begin{cases} (-1)^{|A|+1} \sum_{C \supseteq A} m_b(C) & A \neq \emptyset \\ 0 & A = \emptyset. \end{cases} \tag{13}$$

As b.p.a. do, basic plausibility assignments *meet the normalization constraint*, in other words pl.f. are *normalized sum functions*[1]. As a matter of fact

$$\sum_{A \subseteq \Theta} \mu_b(A) = -\sum_{\emptyset \subsetneq A \subseteq \Theta} (-1)^{|A|} \sum_{C \supseteq A} m_b(C) = -\sum_{C \subseteq \Theta} m_b(C) \cdot \sum_{\emptyset \subsetneq A \subseteq C} (-1)^{|A|} = 1$$

since $-\sum_{\emptyset \subsetneq A \subseteq C} (-1)^{|A|} = -(0 - (-1)^0) = 1$ for Newton's binomial.
However, $\mu_b(A)$ is not always positive.

4.1.1. *Example of b.pl.a.*

Let us consider as an example a b.f. b on the binary frame $\Theta_2 = \{x, y\}$ with b.p.a.

$$m_b(x) = \tfrac{1}{3}, \;\; m_b(\Theta) = \tfrac{2}{3}.$$

The corresponding pl. vector in $\mathbb{R}^{N-2}$ is

$$pl_b = [1 - b(\{x\}^c), 1 - b(\{y\}^c)]' = [1, 2/3]'.$$

Using Equation (10) we can compute its b.pl.a. as

$$\begin{aligned} \mu_b(x) &= (-1)^{|x|+1} \sum_{C \supseteq x} m_b(C) = (-1)^2 \cdot (m_b(x) + m_b(\Theta)) = 1, \\ \mu_b(y) &= (-1)^{|y|+1} \sum_{C \supseteq y} m_b(C) = (-1)^2 \cdot m_b(\Theta) = 2/3, \\ \mu_b(\Theta) &= (-1)^{|\Theta|+1} \sum_{C \supseteq \Theta} m_b(C) = (-1) \cdot m_b(\Theta) = -2/3 < 0 \end{aligned}$$

confirming that b.pl.a. meet the normalization but not the positivity constraint.

By Equation (11) we can infer that, geometrically, each plausibility vector lies on the affine subspace generated by the basis belief functions $\{b_A\}$, with affine coordinates given by the basic plausibility assignment.

4.2. *Plausibility space*

Analogously to what was done for the space of belief functions, we can call *plausibility space* the region $\mathcal{PL}$ of $\mathbb{R}^{N-1}$ whose points correspond to admissible plausibility functions.
The above results on the nature of sum functions of pl.f. allows us to prove that

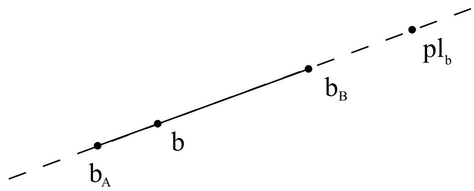

Fig. 4. While each belief functions b lives in the convex region (simplex) determined by the basis b.f. b_A, the corresponding plausibility function pl_b lies in the affine space generated by the basis belief functions. In the case of two basis points b_A, b_B, b and pl_b are points of the segment or the line generated by b_A, b_B respectively.

Theorem 3. *$\mathcal{PL}$ is a simplex*

$$\mathcal{PL} = Cl(pl_A, \emptyset \subsetneq A \subseteq \Theta),$$

whose vertices are given by

$$pl_A = -\sum_{\emptyset \subsetneq B \subseteq A} (-1)^{|B|} b_B. \tag{14}$$

Proof. We just need to re-assemble expression (11) as a convex combination of points, getting (by means of Equation (13))

$$\begin{array}{c} pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} \mu_b(A) b_A = \sum_{\emptyset \subsetneq A \subseteq \Theta} (-1)^{|A|+1} \Big(\sum_{C \supseteq A} m_b(C) \Big) b_A = \\ = \sum_{\emptyset \subsetneq A \subseteq \Theta} (-1)^{|A|+1} b_A \Big(\sum_{C \supseteq A} m_b(C) \Big) = \\ = \sum_{\emptyset \subsetneq C \subseteq \Theta} m_b(C) \Big(\sum_{\emptyset \subsetneq A \subseteq C} (-1)^{|A|+1} b_A \Big) = \sum_{\emptyset \subsetneq C \subseteq \Theta} m_b(C) pl_C \end{array} \tag{15}$$

which is indeed a convex combination since basic probability assignments are non-negative (but $m_b(\emptyset = 0)$) and have unitary sum. Accordingly,

$$\begin{array}{c} \mathcal{PL} = \{pl_b, b \in \mathcal{B}\} = \Big\{ \sum_{\emptyset \subsetneq C \subseteq \Theta} m_b(C) pl_C, \sum_C m_b(C) = 1, m_b(C) \geq 0 \; \forall C \subseteq \Theta \Big\} = \\ = Cl(pl_A, \emptyset \subsetneq A \subseteq \Theta) \end{array}$$

(after exchanging C with A to keep the notation homogeneous). □

It is easy to note that $pl_x = -(-1)^{|x|} b_x = b_x \; \forall x \in \Theta$, so that $\mathcal{B} \cap \mathcal{PL} \supset \mathcal{P}$.

The vertices of the plausibility space have a natural interpretation.

Theorem 4. *The vertex pl_A of the plausibility space is the plausibility vector associated with the basis belief function b_A,*

$$pl_A = pl_{b_A}.$$

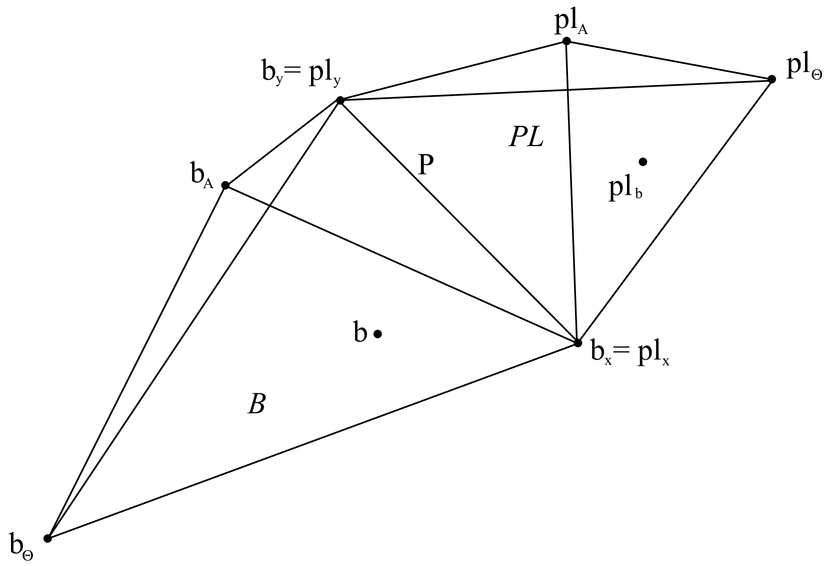

Fig. 5. Convex geometry of the plausibility space $\mathcal{PL}$. Belief space and plausibility space share the probabilistic subspace $\mathcal{P}$, and their vertices b_A, pl_A represent the lower and upper probabilities induced by the same "certain" evidence A.

Proof. Expression (14) is equivalent to

$$pl_A(C) = -\sum_{\emptyset \subsetneq B \subseteq A} (-1)^{|B|} b_B(C) \qquad \forall C \subseteq \Theta.$$

But since $b_B(C) = 1$ if $C \supseteq B$, 0 otherwise, we have that

$$pl_A(C) = -\sum_{B \subseteq A, B \subseteq C, B \neq \emptyset} (-1)^{|B|} = -\sum_{\emptyset \subsetneq B \subseteq A \cap C} (-1)^{|B|}.$$

Now, if $A \cap C = \emptyset$ there is no addenda in the above sum, which goes to zero. Otherwise, for Newton's binomial, we have $pl_A(C) = -\{[1 + (-1)]^{|A \cap C|} - (-1)^0\} = 1$. On the other side, by definition of upper probability,

$$pl_{b_A}(C) = \sum_{B \cap C \neq \emptyset} m_{b_A}(B) = \begin{cases} 1 & A \cap C \neq \emptyset \\ 0 & A \cap C = \emptyset \end{cases}$$

and the two quantities coincide. □

4.3. *Running example: binary case*

To understand these results it can be useful to go back to the simple case study of a binary frame $\Theta_2 = \{x, y\}$, and relate the structures of $\mathcal{B}$ and $\mathcal{PL}$ there. As $b(\Theta) = pl_b(\Theta) = 1$ for all b.f. b, we can neglect as usual the coordinate v_Θ and think of $\mathcal{B}$ as a region of $\mathbb{R}^{N-1}$.

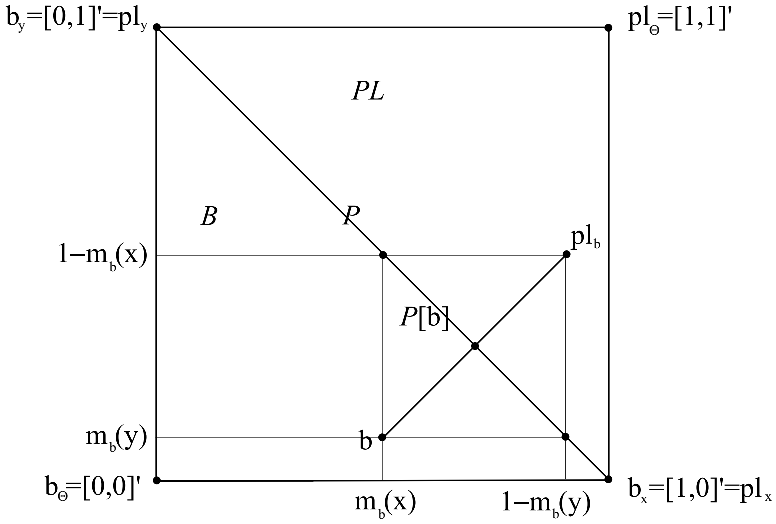

Fig. 6. Geometry of belief and plausibility spaces in the binary case.

Again, $b_\Theta = \mathbf{0} = [0, \cdots, 0]'$ and $pl_\Theta = \mathbf{1} = [1, \cdots, 1]'$ as for the vacuous b.f. $b_\Theta(A) = 0, pl_\Theta(A) = 1$ for all $A \neq \Theta$.
Figure 6 shows the geometry of belief and plausibility spaces for a binary frame $\Theta_2 = \{x, y\}$, where belief and plausibility vectors are points of a plane with coordinates

$$b = [b(x) = m_b(x), b(y) = m_b(y)]'$$
$$pl_b = [pl_b(x) = 1 - m_b(y), pl_b(y) = 1 - m_b(x)]'$$

respectively. These two simplices are symmetric with respect to $\mathcal{P}$,

$$\mathcal{B} = Cl(b_\Theta = \mathbf{0}, b_x, b_y) \qquad \mathcal{PL} = Cl(pl_\Theta = \mathbf{1}, pl_x = b_x, pl_y = b_y)$$

and congruent, so that they can be moved onto each other by means of a rigid transformation.
In this simple case, this rigid transformation is just a reflection through the Bayesian segment $\mathcal{P}$. From Figure 6 it is clear how each individual pair of functions (b, pl_b) determines a line which is orthogonal to $\mathcal{P}$, on which they lay on symmetric positions on the two sides of the Bayesian subspace.

4.4. *Plausibility space in the u.b.f. case*

It is interesting to consider the case of unnormalized belief functions also.
As a matter of fact, it can be seen that Theorems 2 and 4 fully retain their validity. However in the case of Theorem 3, as $m_b(\emptyset) \neq 0$ in general, we need to modify Equation (15) by adding a term related to the empty set, yielding

$$pl_b = \sum_{\emptyset \subsetneq C \subseteq \Theta} m_b(C) pl_C + m_b(\emptyset) pl_\emptyset$$

where $pl_C, C \neq \emptyset$ is still given by Equation (14), and $pl_\emptyset = \mathbf{0}$ is the origin of $\mathbb{R}^N$. Note in fact that even in the u.b.f. case in Equation (14) the empty set is not consider, for $\mu(\emptyset) = 0$.

5. Convex geometry of commonality functions

As we are going to prove in this section, commonality functions are also sum functions and possess some interesting similarities with plausibility functions. However, they present some peculiarity we need to take care of.
We have seen that belief and plausibility functions are such that

$$b(\emptyset) = pl_b(\emptyset) = 0,\ \ b(\Theta) = pl_b(\Theta) = 1;$$

in other words, both b and pl_b can in fact be represented by vectors with $N-2$ coordinates in the hyperplane $\{v : v_\emptyset = 0, v_\Theta = 1\}$.

On the other side, by definition of commonality function,

$$Q_b(\emptyset) = \sum_{A \supseteq \emptyset} m_b(A) = \sum_{A \subseteq \Theta} m_b(A) = 1,\ \ Q_b(\Theta) = \sum_{A \supseteq \Theta} m_b(A) = m_b(\Theta)$$

so that Q_b does not live in general in the hyperplane $\{v : v_\emptyset = 0, v_\Theta = 1\}$, and needs N coordinates to be represented (even though the dimension of $\mathcal{Q}$ is still $N-2$). The geometric counterpart of a plausibility function is then the vector of $\mathbb{R}^N = \mathbb{R}^{2^{|\Theta|}}$

$$Q_b = \sum_{\emptyset \subseteq A \subseteq \Theta} Q_b(A) X_A$$

where $\{X_A : \emptyset \subseteq A \subseteq \Theta\}$ is the extended reference frame introduced in the case of u.b.f. ($A = \Theta, \emptyset$ this time included).

Let us first go back to the simple binary example. As we have just mentioned, $\mathcal{Q}_2$ needs $N = 2^2 = 4$ coordinates to be represented. We have indeed

$$\begin{array}{rl} Q_b(\emptyset) = 1, \quad\ Q_b(x) = & \displaystyle\sum_{A \supseteq \{x\}} m_b(A) = pl_b(x), \\ Q_b(\Theta) = m_b(\Theta),\ Q_b(y) = & \displaystyle\sum_{A \supseteq \{y\}} m_b(A) = pl_b(y) \end{array}$$

and the commonality space $\mathcal{Q}_2$ can be drawn (if we neglect the coordinate $Q_b(\emptyset)$ which is constant $\forall b$) as in Figure 7.

5.1. *Comm.f. as sum functions: Basic commonality assignment*

To understand the geometry of comm.f. in the general case we need as before to express Q_b as a sum function. We can use Theorem 1 to change the coordinate base and get the coordinates of Q_b with respect to the base $\{b_A, \emptyset \subseteq A \subseteq \Theta\}$ formed by all the basis unnormalized belief functions. We have

$$\begin{array}{c} Q_b = \displaystyle\sum_{\emptyset \subseteq A \subseteq \Theta} Q_b(A) \Big(\sum_{B \supseteq A} b_B (-1)^{|B \setminus A|} \Big) = \\ = \displaystyle\sum_{\emptyset \subseteq B \subseteq \Theta} b_B \Big(\sum_{A \subseteq B} (-1)^{|B \setminus A|} Q_b(A) \Big) = \sum_{\emptyset \subseteq B \subseteq \Theta} q_b(B) b_B \end{array}$$

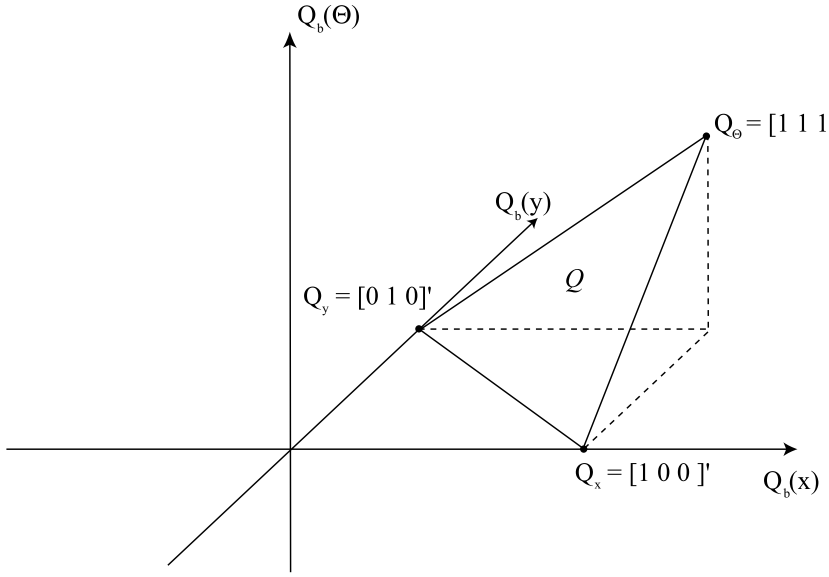

Fig. 7. Commonality space in the binary case.

i.e. Q_b is a sum function with Moebius inverse

$$\begin{array}{l} q_b : 2^\Theta \rightarrow [0,1] \\ \quad B \;\mapsto q_b(B) = \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B \setminus A|} Q_b(A) \end{array}$$

which we can call *basic commonality assignment* (b.comm.a.).

To compute the explicit form of q_b we can write

$$\begin{array}{l} q_b(B) = \sum_{\emptyset \subseteq A \subseteq B} (-1)^{|B\setminus A|} \big(\sum_{C \supseteq A} m_b(C) \big) = \sum_{\emptyset \subsetneq A \subseteq B} (-1)^{|B\setminus A|} \big(\sum_{C \supseteq A} m_b(C) \big) + \\ +(-1)^{|B|-|\emptyset|} \sum_{C \supseteq \emptyset} m_b(C) = \sum_{B \cap C \neq \emptyset} m_b(C) \big(\sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B \setminus A|} \big) + (-1)^{|B|}. \end{array}$$

But now, since $B \setminus A = B \setminus C + B \cap C \setminus A$, we have that

$$\begin{array}{l} \sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B\setminus A|} = (-1)^{|B \setminus C|} \sum_{\emptyset \subsetneq A \subseteq B \cap C} (-1)^{|B \cap C| - |A|} = \\ = (-1)^{|B \setminus C|} \Big[(1-1)^{|B \cap C|} - (-1)^{|B \cap C| - |\emptyset|} \Big] = (-1)^{|B|+1} \end{array}$$

so that the b.comm.a. $q_b(B)$ can be expressed as

$$\begin{array}{l} q_b(B) = (-1)^{|B|+1} \sum_{B \cap C \neq \emptyset} m_b(C) + (-1)^{|B|} = (-1)^{|B|} (1 - \sum_{B \cap C \neq \emptyset} m_b(C)) = \\ \quad = (-1)^{|B|} (1 - pl_b(B)) = (-1)^{|B|} b(B^c) \end{array} \qquad (16)$$

(note that $q_b(\emptyset) = (-1)^{|\emptyset|} b(\emptyset) = 1$).

5.2. *Comm.f. as non-normalized sum functions*

The basic commonality assignment does not meet the normalization axiom, as

$$\sum_{\emptyset\subseteq B\subseteq\Theta} q_b(B) = Q_b(\Theta) = m_b(\Theta).$$

In other words, whereas belief functions are normalized sum functions (n.s.f.) with non-negative Moebius inverse, and plausibility functions are normalized sum functions, commonality functions are sum functions but *not* n.s.f.

Going back to the example of Section 4.1.1, the b.comm.a. associated with $m_b(x) = 1/3$, $m_b(\Theta) = 2/3$ is (by Equation (16))

$$q_b(\emptyset) = (-1)^{|\emptyset|} b(\Theta) = 1, \qquad q_b(x) = (-1)^{|x|} b(y) = -m_b(y) = 0,$$
$$q_b(y) = (-1)^{|y|} b(x) = -m_b(x) = -1/3, \qquad q_b(\Theta) = (-1)^{|\Theta|} b(\emptyset) = 0$$

so that

$$\sum_{\emptyset\subseteq B\subseteq\Theta} q_b(B) = 1 - 1/3 = 2/3 = m_b(\Theta) = Q_b(\Theta).$$

5.3. *Commonality space*

We can use the b.comm.a. (16) to recover the shape of the space $\mathcal{Q} \subset \mathbb{R}^N$ of all commonality functions. We have

$$Q_b = \sum_{\emptyset\subseteq B\subseteq\Theta} (-1)^{|B|} b_B \Big(\sum_{\emptyset\subseteq A\subseteq B^c} m_b(A) \Big) = \sum_{\emptyset\subseteq A\subseteq\Theta} m_b(A) \Big(\sum_{\emptyset\subseteq B\subseteq A^c} (-1)^{|B|} b_B \Big) =$$
$$= \sum_{\emptyset\subseteq A\subseteq\Theta} m_b(A) Q_A$$

where

$$Q_A \doteq \sum_{\emptyset\subseteq B\subseteq A^c} (-1)^{|B|} b_B \qquad (17)$$

is the A-the vertex of the *commonality space*, which is hence given by

$$\mathcal{Q} = Cl(Q_A, \emptyset \subseteq A \subseteq \Theta).$$

Again, Q_A is the commonality function associated with the basis belief function b_A:

$$Q_{b_A} = \sum_{\emptyset\subseteq B\subseteq\Theta} q_{b_A}(B) b_B$$

where $q_{b_A}(B) = (-1)^{|B|}$ if $B^c \supseteq A$ i.e. $B \subseteq A^c$, while $q_{b_A}(B) = 0$ otherwise, Hence

$$Q_{b_A} = \sum_{\emptyset\subseteq B\subseteq A^c} (-1)^{|B|} b_B = Q_A$$

and the two quantities coincide.

In the binary case, for example, the vertices of $\mathcal{Q}_2$ are, according to Equation (17)

$$\begin{aligned} Q_\emptyset &= \sum_{\emptyset\subseteq B\subseteq\Theta} (-1)^{|B|} b_B = b_\emptyset + b_\Theta - b_x - b_y = \\ &= [1111]' + [0001]' - [0101]' - [0011]' = [1000]' = Q_{b_\emptyset}, \\ Q_x &= \sum_{\emptyset\subseteq B\subseteq\{y\}} (-1)^{|B|} b_B = b_\emptyset - b_y = [1111]' - [0011]' = [1100]' = Q_{b_x} \end{aligned}$$

etcetera.

6. Congruence

Analogously to belief functions, both plausibility and commonality functions can then be thought of as sum functions on the partially ordered set 2^Θ (even though whereas b.f. and pl.f. are normalized sum functions, comm.f. are not). This in turn allows to describe them as points of some simplices $\mathcal{B}, \mathcal{PL}$ and $\mathcal{Q}$ in the Cartesian space.

We have also seen that, in case of a binary frame of discernment, $\mathcal{B}$ and $\mathcal{PL}$ are congruent, i.e. they can be superposed by means of a rigid transformation. We may wonder if those features are characteristic of the binary case only, or can be extended and proven in the general case. It is also natural to suppose that a similar relation exists between belief and plausibility spaces and the commonality space.

Unfortunately, it turns out that reflection is too simple a relation to hold for an arbitrary belief space. However, we will see that congruence is nevertheless a general property, and reflection is a particular case of a more general rigid transformation.

6.1. *Congruence of belief and plausibility spaces*

Let us then first consider the general relation between belief and plausibility simplices.

Theorem 5. *The corresponding 1-dimensional sides $Cl(b_A, b_B)$ and $Cl(pl_A, pl_B)$ of belief and plausibility spaces are congruent, namely*

$$\|pl_B - pl_A\|_p = \|b_A - b_B\|_p$$

where $\|\|_p$ denotes the classical norm $\|\mathbf{v}\|_p \doteq \sqrt{\sum_{i=1}^N |v_i|^p}$, for all $p = 1, 2, ..., +\infty$.

Proof. This a direct consequence of the definition of plausibility function. Let us denote with C, D two generic subsets of Θ. As $pl_A(C) = 1 - b_A(C^c)$ we have $b_A(C^c) = 1 - pl_A(C)$, which implies

$$b_A(C^c) - b_B(C^c) = 1 - pl_A(C) - 1 + pl_B(C) = pl_B(C) - pl_A(C).$$

This in turn means

$$\sum_{C\subset\Theta} |pl_B(C) - pl_A(C)|^p = \sum_{C\subset\Theta} |b_A(C^c) - b_B(C^c)|^p = \sum_{D\subset\Theta} |b_A(D) - b_B(D)|^p$$

for all p. □

Notice that the proof of Theorem 5 holds no matter if the couple $(\emptyset, \emptyset^c = \Theta)$ is considered or not. A straightforward implication is then that

Corollary 3. *$\mathcal{B}$ and $\mathcal{PL}$ are congruent.*

but also

Corollary 4. *$\mathcal{B}^U$ and $\mathcal{PL}^U$ are congruent.*

since their corresponding 1-dimensional faces have the same length, because of the generalization of a well-known Euclid's theorem stating that triangles with sides of the same length are congruent. It is worth to notice that, although this holds for *simplices* (generalized triangles), the same is not true for *polytopes* in general, i.e. convex closures of a number of vertices greater than $n+1$ where n is the dimension of the Cartesian space in which they are defined (think of a square and a rhombus with sides of length 1).

6.1.1. *Binary case*

Let us then see what happens to belief, plausibility and commonality spaces in the case of unnormalized belief functions defined on a binary frame. All three simplices have $N = 2^{|\Theta|}$ vertices and dimension $N-1$:

$$\mathcal{B}^U = Cl(b_A, \emptyset \subseteq A \subseteq \Theta),\ \mathcal{PL}^U = Cl(pl_A, \emptyset \subseteq A \subseteq \Theta),\ \mathcal{Q}^U = Cl(Q_A, \emptyset \subseteq A \subseteq \Theta).$$

In the binary case they form three-dimensional polytopes immersed in a four-dimensional Cartesian space:

$$\begin{array}{c} \mathcal{B} = Cl(b_\emptyset = [1\ 1\ 1\ 1]', b_x = [0\ 1\ 0\ 1]', b_y = [0\ 0\ 1\ 1]', b_\Theta = [0\ 0\ 0\ 1]') \\ \mathcal{PL} = Cl(pl_\emptyset = [0\ 0\ 0\ 0]', pl_x = [0\ 1\ 0\ 1]', pl_y = [0\ 0\ 1\ 1]', pl_\Theta = [0\ 1\ 1\ 1]') \\ \mathcal{Q} = Cl(Q_\emptyset = [1\ 0\ 0\ 0]', Q_x = [1\ 1\ 0\ 0]', Q_y = [1\ 0\ 1\ 0]', Q_\Theta = [1\ 1\ 1\ 1]') \end{array} \tag{18}$$

We already know that $\mathcal{PL}_2$ and $\mathcal{B}_2$ are congruent. By Equation (18) it follows that

$$\begin{array}{c} \|b_\emptyset - b_x\|_2 = \|[1010]'\|_2 = \sqrt{2} = \|[0101]'\|_2 = \|pl_x - pl_\emptyset\|_2, \\ \|b_y - b_\Theta\|_2 = \|[0010]'\|_2 = 1 = \|[0100]'\|_2 = \|pl_\Theta - pl_y\|_2 \end{array}$$

etcetera, and as $\mathcal{B}_2^U$ and $\mathcal{PL}_2^U$ are simplices they are also congruent.

6.2. *Congruence of plausibility and commonality spaces*

A similar result holds for plausibility and commonality spaces. We first need to point out the peculiar relationship which exists between the vertices of plausibility and commonality spaces in the u.b.f case, as

$$pl_A = -\sum_{\emptyset \subsetneq B \subseteq A} (-1)^{|B|} b_B, \qquad \begin{array}{l} Q_A = \displaystyle\sum_{\emptyset \subseteq B \subseteq A^c} (-1)^{|B|} b_B = \sum_{\emptyset \subsetneq B \subseteq A^c} (-1)^{|B|} b_B + b_\emptyset = \\ \qquad = -pl_{A^c} + b_\emptyset. \end{array} \tag{19}$$

Theorem 6. *The 1-dimensional faces $Cl(Q_B, Q_A)$ and $Cl(pl_{B^c}, pl_{A^c})$ of commonality and plausibility spaces respectively are congruent, namely*

$$\|Q_B - Q_A\|_p = \|pl_{B^c} - pl_{A^c}\|_p.$$

Proof. Since $Q_A = b_\Theta - pl_{A^c}$ then $Q_A - Q_B = b_\emptyset - pl_{A^c} - b_\emptyset + pl_{B^c} = pl_{B^c} - pl_{A^c}$ so that the two sides are obviously congruent. □

The following map between vertices of $\mathcal{PL}^U$ and $\mathcal{Q}^U$

$$Q_A \mapsto pl_{A^c} \tag{20}$$

then maps 1-dimensional faces of the commonality space to congruent faces of the plausibility space

$$Cl(Q_A, Q_B) \mapsto Cl(pl_{A^c}, pl_{B^c})$$

and the two simplices are congruent. However, (20) clearly acts as a 1-1 application on *unnormalized* basis commonality and plausibility functions (as the complement of $\emptyset$ is Θ so that $Q_\Theta \mapsto pl_\emptyset$). Therefore we can only claim that

Corollary 5. *$\mathcal{Q}^U$ and $\mathcal{PL}^U$ are congruent.*

6.2.1. *Running example: congruence of $\mathcal{Q}_2$ and $\mathcal{PL}_2$*

Let us get back, for instance, to the binary example. It is easy to see from Figure 7 that $\mathcal{PL}_2$ and $\mathcal{Q}_2$ are *not* congruent in the case of normalized b.f., as $\mathcal{Q}_2$ is an equilateral triangle with sides of length $\sqrt{2}$, while $\mathcal{PL}_2$ has two sides of length 1.

If we instead consider u.b.f. we get, recalling Equation (18),

$$\begin{array}{rr} Q_\Theta - Q_\emptyset = [0\ 1\ 1\ 1]', & pl_\Theta - pl_\emptyset = [0\ 1\ 1\ 1]' \\ Q_x - Q_y = [0\ 1\ -1\ 0]', & pl_x - pl_y = [0\ 1\ -1\ 0]' \\ Q_x - Q_\Theta = [0\ 0\ -1\ -1]', & pl_\emptyset - pl_y = [0\ 0\ -1\ -1]' \end{array} \tag{21}$$

etcetera, confirming that $\mathcal{Q}^U_2$ and $\mathcal{PL}^U_2$ are indeed congruent.

7. Explicit form of the rigid transformation

Belief, plausibility and commonality functions can be seen as points of a large enough Cartesian space. We have seen that they form simplices which can be moved onto each other by means of a rigid transformation. We can however go further, and try and analyze the geometric behavior of *single* functions or of the triplet of associated non-additive measures (b, pl_b, Q_b).

In binary case (Section 4.3) the pointwise geometry of a plausibility vector can be described in terms of a reflection with respect to the probability simplex $\mathcal{P}$. In the general case, as the simplices $\mathcal{B}^U$, $\mathcal{PL}^U$, and $\mathcal{Q}^U$ are all congruent, there must exist an Euclidean transformation $\tau \in E(N)$ mapping one simplex onto the other one. In this final part of the paper we will indeed give an explicit description of the simple

rigid map between $\mathcal{PL}$ and $\mathcal{B}$ in the normalized case, and between $\mathcal{PL}^U$ and $\mathcal{Q}^U$ in the unnormalized case.
As this is nothing but the geometric counterpart of the probabilistic relation between upper and lower probabilities, this completes the geometric semantics of the theory of evidence.

7.1. *Belief and plausibility spaces*

In case of belief and plausibility spaces (in the standard, normalized case) the rigid transformation is clearly encoded by Equation (3):

$$pl_b(A) = 1 - b(A^c).$$

This implies that, since $pl_b = \sum_{\emptyset \subsetneq A \subseteq \Theta} pl_b(A) X_A$,

$$pl_b = \mathbf{1} - b^c$$

where b^c is the unique belief function whose belief values are the same as b's on the complement of each event A,

$$b^c(A) = b(A^c).$$

Besides, as in the normalized case $\mathbf{1} = pl_\Theta$ and $\mathbf{0} = b_\Theta$, this can be written as

$$pl_b = b_\Theta + pl_\Theta - b^c,$$

Geometrically, this means that the segments $Cl(b_\Theta, pl_\Theta)$ have the same barycenter, as

$$\frac{pl_b + b^c}{2} = \frac{b_\Theta + pl_\Theta}{2}.$$

In other words, *the plausibility vector pl_b associated with b is the reflection in $\mathbb{R}^{N-2}$ through the segment $Cl(b_\Theta, pl_\Theta) = Cl(\mathbf{0}, \mathbf{1})$ of the "complement" belief function b^c*. Geometrically, b^c is obtained from b by means of another reflection (swapping the coordinates associated with the axes X_A and X_{A^c}), so that the form of the desired transformation is completely determined. Figure 8 illustrates the nature of the transformation, and its instantiation in the binary case for normalized belief functions.

In the case of u.b.f. $b_\emptyset = \mathbf{1}$, $pl_\emptyset = \mathbf{0}$ so that we have

$$pl_b = pl_\emptyset + b_\emptyset - b^c$$

i.e. pl_b is the reflection of b^c through the segment $Cl(b_\emptyset, pl_\emptyset) = Cl(\mathbf{0}, \mathbf{1})$.

7.2. *Commonality and plausibility spaces*

The transformation is also quite simple in the case of the pair $(\mathcal{PL}^U, \mathcal{Q}^U)$. We can indeed use Equation (19) to determine the geometric relationship between $\mathcal{PL}^U$ and

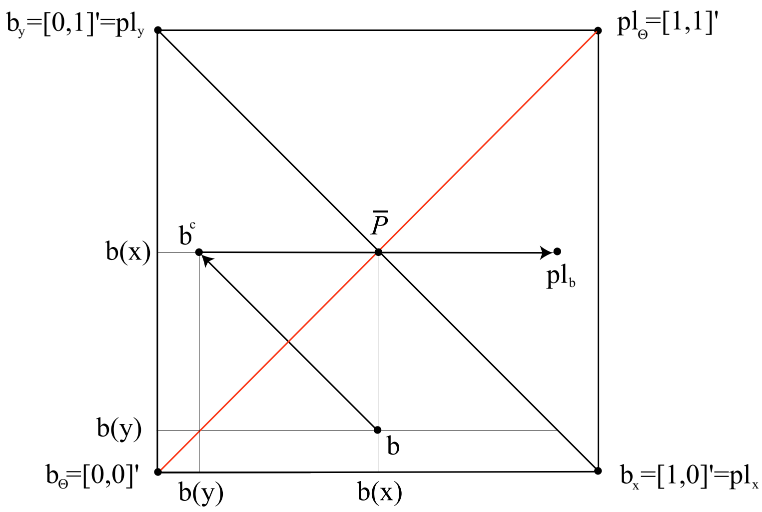

Fig. 8. Rigid transformation mapping b onto pl_b in the normalized case. In the binary case the middle point of the segment $Cl(\mathbf{0},\mathbf{1})$ is the mean probability $\overline{\mathcal{P}}$.

$\mathcal{Q}^U$. As a matter of fact

$$Q_b = \sum_{\emptyset\subseteq A\subseteq\Theta} m_b(A)Q_A = \sum_{\emptyset\subseteq A\subseteq\Theta} m_b(A)(b_\emptyset - pl_{A^c}) = b_\emptyset - \sum_{\emptyset\subseteq A\subseteq\Theta} m_b(A)pl_{A^c} = \\ = b_\emptyset - pl_{b^{m^c}} \tag{22}$$

where b^{m^c} is the unique belief function whose b.p.a. is

$$m_{b^{m^c}}(A) = m_b(A^c).$$

But then, since $pl_\emptyset = \mathbf{0} = [0,\cdots,0]$ for unnormalized belief functions (remember the binary example) we can rewrite Equation (22) as

$$Q_b = pl_\emptyset + b_\emptyset - pl_{b^{m^c}}.$$

Again, this means that *the commonality vector associated with b is the reflection in $\mathbb{R}^N$ through the segment $Cl(pl_\emptyset, b_\emptyset) = Cl(\mathbf{0},\mathbf{1})$ of the plausibility vector $pl_{b^{m^c}}$ associated with the belief function b^{m^c}*.
In this case, though, b^{m^c} is obtained from b by swapping the coordinates with respect to the base $\{b_A, \emptyset \subseteq A \subseteq \Theta\}$. A pictorial representation for the binary case (similar to Figure 8) is more difficult in this case as $\mathbb{R}^4$ is involved.

It is natural to stress the analogy between the two rigid transformations $\tau^U_{\mathcal{B}^U\mathcal{PL}^U} : \mathcal{B}^U \to \mathcal{PL}^U$ and $\tau_{\mathcal{PL}^U\mathcal{Q}^U} : \mathcal{PL}^U \to \mathcal{Q}^U$ mapping respectively a u.b.f. onto its pl.f., and an u.pl.f. onto the corresponding comm.f.:

$$\begin{array}{ccccc} b & \stackrel{b(A)\mapsto b(A^c)}{\longrightarrow} & b^c & \stackrel{refl.\ through\ Cl(\mathbf{0},\mathbf{1})}{\longrightarrow} & pl_b \\ pl_b & \stackrel{m_b(A)\mapsto m_b(A^c)}{\longrightarrow} & b^{m^c} & \stackrel{refl.\ through\ Cl(\mathbf{0},\mathbf{1})}{\longrightarrow} & Q_b. \end{array}$$

They have both the form of a sequence of two reflections: a swap of the axes of the reference frame $\{X_A\}$ ($\{b_A\}$) induced by set-theoretic complement, plus a reflection with respect to the center of the segment $Cl(\mathbf{0}, \mathbf{1})$.

8. Comments and conclusions

Subjective probability and combinatorics are apparently unrelated fields. However belief functions, as they are functions defined on power sets, are inherently related to a number of topics of combinatorics like Boolean algebras, partially ordered sets and lattices[35], matroids[19] just to cite a few of them. These links have never been systematically explored, even though some work has been recently done in this direction [35,34,41,18]. The geometric approach to the theory of evidence, even though originally motivated by the approximation problem, is indeed a step in this perspective. Each element of the theory of evidence can be described through the language of convex geometry.

In this paper, in particular, we extended the geometric approach in order to study the geometric behavior of two other quantities inherently associated with a belief function, i.e. the plausibility and commonality functions. In terms of subjective belief they carry the same evidence as belief functions do. From a combinatorial point of view, they share the form of sum functions on a specific partially ordered set, the power set 2^Θ. We are then allowed to introduce in both cases the analogous of the basic probability assignment, and used it to understand the simplicial form of plausibility and commonality spaces.

In fact, from a combinatorial point of view, b.f., pl.f. and comm.f. (even though they are equivalent representation of the same evidence) form a hierarchy of sum functions whose Moebius inverse meets both normalization and positivity axiom (b.p.a.), only the normalization constraint (b.pl.a.), and none of them (b.comm.a.) respectively.

Quantity	Moebius inverse	
belief function	b.p.a.	non-negative n.s.f.
plausibility function	b.pl.a.	n.s.f.
commonality function	b.comm.a.	sum function.

Nevertheless the related spaces possess a similar convex geometry. Given the intuition provided by the case of binary frames we analyzed the global structure of those simplices, by proving in particular the congruence of $\mathcal{B}$ and $\mathcal{PL}$ (in the case of both standard and unnormalised belief functions), and of the pair $\mathcal{PL}^U, \mathcal{Q}^U$ (for u.b.f. only). We ventured into the description of the point-wise geometry of the triplet (b, pl_b, Q_b), showing that it is described by a sequence of two reflections with respect to the two reference frames associated with belief values and mass values. This places another element in the picture of the relationship between evidential formalism and discrete mathematics, and its implications in terms of the meaning of the evidential machinery itself. Belief functions (lower probabilities) are points of a simplex in which basic probability assignment plays the role of convex coordinate, while Dempster's rule itself is nothing but an intersection of linear spaces[17] and it can be conjectured this holds for other combination rules too. Now we know that upper probabilities too (pl.f.) have a simplicial geometry, and the dual relation between upper and lower probabilities is in fact a rigid transformation. It may well be that the Dempster-Shafer formalism is nothing but some cryptomorphic geometric calculus.

Points of contact with the field of geometric probability[42] (which studies invariant measures on sets of geometric objects and relates them to additive probability measures) are worth to study, as they can lead to a fertile contamination of the two fields. For instance, a well-known result[42] states that all simplicial complexes on a partially ordered set form a distributive lattice. As we know that belief, plausibility and commonality functions, but also finite possibility measures form simplicial complexes in 2^Θ (as a simplex is just a principal complex) this could eventually bring to an algebraic interpretation of the mutual relations between all those measures which mirrors the geometric one.

Appendix A. Proofs

A.1. *Proof of Theorem 1*

We first need to notice that a basis b.f. can be expressed as

$$b_A = \sum_{C \supseteq A} X_C. \tag{A.1}$$

If (9) is true we have that

$$b_A = \sum_{C \supseteq A} X_C = \sum_{C \supseteq A} \sum_{B \supseteq C} b_B (-1)^{|B-C|} = \sum_{B \supseteq A} b_B \Big(\sum_{A \subseteq C \subseteq B} (-1)^{|B-C|} \Big).$$

Let us then consider the factor $\sum_{A \subseteq C \subseteq B} (-1)^{|B-C|}$. When $A = B$ then $C = A = B$ and the coefficient becomes 1. On the other side, when $B \neq A$ we have

$$\sum_{A \subseteq C \subseteq B} (-1)^{|B-C|} = \sum_{D \subseteq B \setminus A} (-1)^D = 0$$

for Newton's binomial $\sum_{k=0}^n 1^{n-k}(-1)^k = 0$. Hence $b_A = b_A$.

A.2. *Proof of Theorem 2*

The definition (3) of upper probability yields

$$\mu_b(A) = \sum_{B\subseteq A}(-1)^{|A\setminus B|}pl_b(B) = \sum_{B\subseteq A}(-1)^{|A\setminus B|}(1-b(B^c)) = \sum_{B\subseteq A}(-1)^{|A\setminus B|} + \\ -\sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B^c) = 0 - \sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B^c) = -\sum_{B\subseteq A}(-1)^{|A\setminus B|}b(B^c)$$

since for Newton's binomial $\sum_{B\subseteq A}(-1)^{|A\setminus B|} = 0$ if $A \neq \emptyset$, $(-1)^{|A|}$ otherwise. If $B \subseteq A$ then $B^c \supseteq A^c$, so that the above expression becomes

$$-\sum_{\emptyset\subsetneq B\subseteq A}(-1)^{|A\setminus B|}\Big(\sum_{C\subseteq B^c} m_b(C)\Big) = -\sum_{C\subseteq\Theta} m_b(C)\Big(\sum_{B:B\subseteq A, B^c\supseteq C}(-1)^{|A\setminus B|}\Big) = \\ = -\sum_{C\subseteq\Theta} m_b(C)\Big(\sum_{B\subseteq A\cap C^c}(-1)^{|A\setminus B|}\Big) \tag{A.2}$$

for $B^c \supseteq C, B \subseteq A$ is equivalent to $B \subseteq C^c, B \subseteq A \equiv B \subseteq (A \cap C^c)$.
Let us now analyze the function of C

$$f(C) \doteq \sum_{B\subseteq A\cap C^c}(-1)^{|A\setminus B|}.$$

If $A \cap C^c = \emptyset$ then $B = \emptyset$ and the sum is $(-1)^{|A|}$. If $A \cap C^c \neq \emptyset$, instead, we can write $D \doteq C^c \cap A$ and obtain

$$f(C) = \sum_{B\subseteq D}(-1)^{|A\setminus B|} = \sum_{B\subseteq D}(-1)^{|A\setminus D|+|D\setminus B|} =$$

since $B \subseteq D \subseteq A$ and $|A| - |B| = |A| - |D| + |D| - |B|$,

$$= (-1)^{|A|-|D|}\cdot\sum_{B\subseteq D}(-1)^{|D|-|B|} = 0$$

given that $\sum_{B\subseteq D}(-1)^{|D|-|B|} = 0$ for Newton's binomial again. Eventually

$$f(C) = \begin{cases} 0 & C^c \cap A \neq \emptyset \\ (-1)^{|A|} & C^c \cap A = \emptyset \end{cases}$$

and we can rewrite expression (A.2) as

$$-\sum_{C\subseteq\Theta} m_b(C) f(C) = -\sum_{C:C^c\cap A\neq\emptyset} m_b(C)\cdot 0 - \sum_{C:C^c\cap A=\emptyset} m_b(C)\cdot(-1)^{|A|} = \\ = (-1)^{|A|+1}\sum_{C:C^c\cap A=\emptyset} m_b(C) = (-1)^{|A|+1}\sum_{C\supseteq A} m_b(C).$$

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