An Analytic Solution for the Perspective 4-Point Problem

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The perspective *n*-point (P*n*P) problem is the problem of finding the position and orientation of a camera with respect to a scene object from *n* correspondence points. In this paper we propose an analytic solution for the perspective 4-point problem. The solution is found by replacing the four points with a pencil of three lines and by exploring the geometric constraints available with the perspective camera model. We show how the P4P problem is cast into the problem of solving a biquadratic polynomial equation in one unknown. Although developed as part of an object recognition from a single view system [6], the solution might well be used for hand–eye coordination, landmark-guided navigation, and for fast determination of exterior camera parameters in general. © 1989 Academic Press, Inc.

1. INTRODUCTION

One of the fundamental goals of computer vision is to discover properties that are intrinsic to a scene by analyzing one or several images of this scene. Within this paradigm, an essential process is the determination of the position and orientation of the sensing device (the camera) with respect to objects in the scene. This problem is known as the *exterior camera calibration problem* and it has many interesting applications in robotics and cartography. Important applications in robotics are: sensor calibration [12], object recognition and localization from a single view [6, 9, 2], stereo sensor calibration [3], hand/eye coordination [13], and sensory based navigation [1].

In cartography the problem is to determine the location in space from which an image or a photograph was obtained by recognizing a set of landmarks appearing in the image [4].

More formally, the problem may be stated as follows: Given a set of points with their coordinates in an object-centered frame and their corresponding projections onto an image plane and given the intrinsic camera parameters, find the transformation matrix (three rotations and three translations) between the object frame and the camera frame (see Fig. 1).

This problem is referred to as the *perspective n-point problem* and is usually solved using least-squares techniques. An elegant least-squares solution has recently been proposed [9]. Least-squares techniques require the computation of numerical solutions. For these solutions to be stable a large set of data points are needed which inherently augments the complexity of the computation. This is not desirable especially when the computation resides in the inner loop of a recognition/localization process [6, 9, 5, 11, 7, 8, 2].
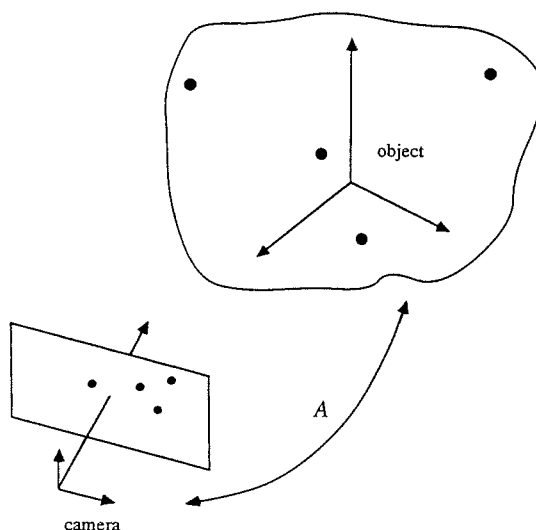
FIG. 1. The transformation matrix between an object-centered frame and a camera-centered frame.

For these reasons a certain number of researchers have tried to determine the minimum number of points necessary to find a solution, and associated with each set of points they have tried to find a closed form expression of the problem. A finite number of solutions is available only when the number of points is equal or greater than 3. The following is a brief review of the suggested solutions.

• *Three points*. Rives *et al.* [10] derive a set of three quadratic equations with three unknowns. These unknowns are the distances from the optical center of the camera to the three points. In theory there are eight solutions. Fischler and Bolles [4] notice that for every real positive solution there is a real negative solution and hence a maximum of four solutions are in fact possible. They derive a closed form expression, namely, a biquadratic polynomial in one unknown.

• *Four points*. When the points are coplanar a set of solutions can be found by considering any three among the four points and verifying the solution with the fourth point: A unique solution is thus found in [4]. When the points are not coplanar a closed form expression does not appear to have been derived. Rives *et al.* [10] solve a set of six quadratic equations with four unknowns. Fischler and Bolles [4] attack the problem by finding solutions associated with subsets of three points and selecting the solutions that they have in common. They provide a geometric construction which shows that unlike the coplanar case, two solutions may be available.

• *Five points*. For five points in general position the strategy mentioned above can as well be applied: Compare the solutions obtained with subsets of three or four points.

• *Six points* or more will always produce a unique solution: For six points we obtain 12 linear equations which are enough to determine the 12 coefficients of the transformation matrix, nine for the rotations and three for the translations.
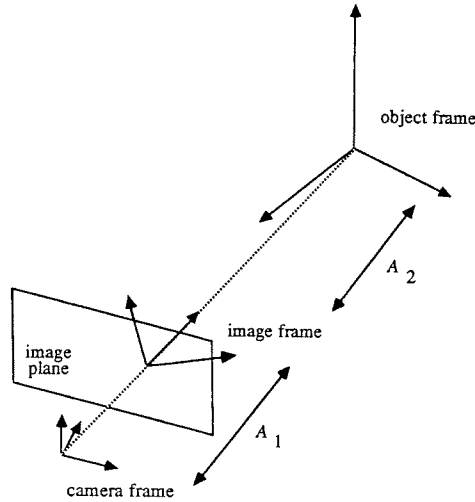
FIG. 2. The decomposition of the transformation matrix $\mathbf{A}$ into two matrices: $\mathbf{A}_1$ which maps the image frame into the camera frame and $\mathbf{A}_2$ which maps the object frame into the image frame.

- *Three lines*. An alternative solution is to use lines instead of points. Horaud [6] suggests a constructive method for the case of a pencil of three non-coplanar lines. Dhome *et al*. [2] solve for the general case of three arbitrary lines. The solutions are given by the roots of a polynomial of order 8 in one unknown.

In this paper we derive an analytic solution for the case of four non-coplanar points, namely a biquadratic polynomial in one unknown. Roots of such an equation can be found in closed form or by an iterative method. Finding a solution for four non-coplanar points is equivalent to finding a solution for a pencil of three non-coplanar lines: The three lines share one of the four points. Notice that these lines may or may not correspond to physical linear edges in the scene. Finding a closed form solution for four non-coplanar points is important for several reasons. First, they provide fewer solutions than three points. Second, the solutions are more stable when the points are not coplanar, because they do not depend on the relative orientation of the image plane with respect to the scene plane containing the points. Third, the computation of such a solution is very fast and therefore it can be included in a runtime visual process.

2. THE SOLUTION

In order to compute the transformation matrix $\mathbf{A}$ of Fig. 1 we decompose it into two matrices, $\mathbf{A}_1$ and $\mathbf{A}_2$, and we define three frames; a *camera-centered frame*, an *image-centered frame*, and an *object-centered frame*, e.g., Fig. 2. $\mathbf{A}_1$ is the transformation matrix from the image frame to the camera frame and $\mathbf{A}_2$ is the transformation matrix from the object frame (Fig. 3) to the image frame. Therefore we have:

$$\mathbf{A} = \mathbf{A}_1\mathbf{A}_2. \tag{1}$$

The four non-coplanar points are replaced by a pencil of three non-coplanar line
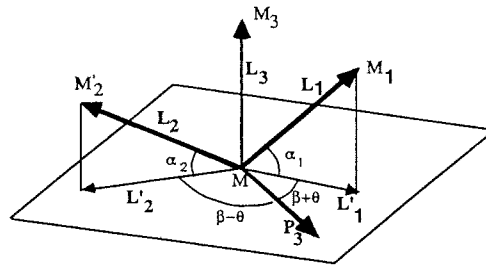
FIG. 3. An "object frame" is associated with the four points.

segments, as shown on Fig. 4. We denote by L_1, L_2, and L_3 the unit vectors associated with the directions of these lines and let l_1, l_2, and l_3 be the unit vectors associated with their projections onto the image plane. The three frames used throughout the paper are defined as follows:

- *The object coordinate system* is defined as follows (see Fig. 3). L_3 is the x-axis. Let P_3 be a unit vector in the plane perpendicular to L_3. The geometric meaning of P_3 will soon be made clear. Let P_3 be the y-axis of the object frame; the
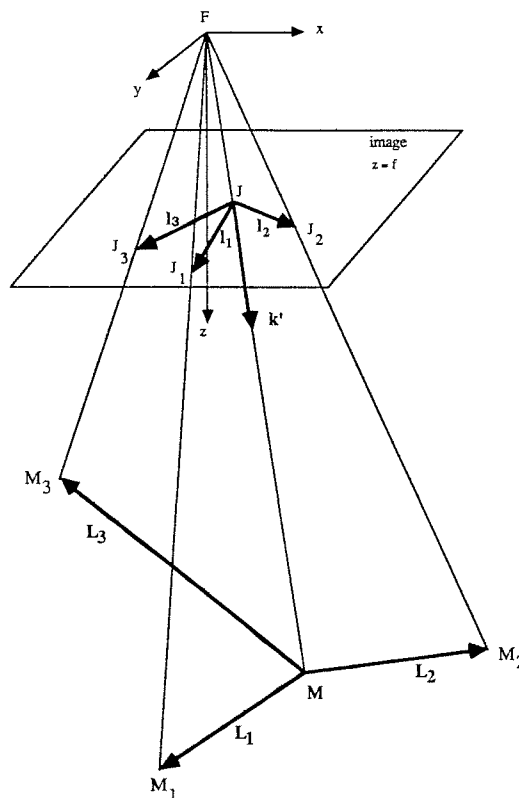
FIG. 4. The geometry of the 4-point perspective problem.

z-axis is defined by the cross product $L_3 \times P_3$. It is easy to determine the object frame coordinates of L_1 and L_2, e.g., Fig. 3:

$$L_1 = \sin \alpha_1 L_3 + \cos \alpha_1 \cos(\beta + \theta) P_3 - \cos \alpha_1 \sin(\beta + \theta) L_3 \times P_3 \qquad (2)$$

$$L_2 = \sin \alpha_2 L_3 + \cos \alpha_2 \cos(\beta - \theta) P_3 + \cos \alpha_2 \sin(\beta - \theta) L_3 \times P_3. \qquad (3)$$

In these formulas α_1 is the value of the angle between L_1 and L_1' (the projection of L_1 onto the yz-plane). The value of the angle between L_1' and L_2' is 2β. Notice that in these formulas θ is an unknown which will be determined.

- *The camera coordinate system* has its origin at the focal point F. The z-axis is the optical axis of the camera and the xy-plane is parallel to the image plane. The image is at distance f (the focal length) from the origin along the z-axis.

- We define now an *image coordinate system* which is rigidly attached to the projections of the object features, i.e., the image features. The projection of L_1 onto the image plane is l_1. The focal point F and l_1 define a plane called the *interpretation plane*. Notice that this plane is rigidly attached to the camera frame since the coordinates of l_1 are *measured* in this frame. All the spatial interpretations of l_1, and hence L_1, belong to this plane. Let P_1 be the normal unit vector associated with this plane. Hence P_3 mentioned above is the unit vector normal to the interpretation plane associated with l_3. Let J be the point of intersection of the image lines l_1, l_2, and l_3.

We are now ready to define the image coordinate system. The x-axis lies along the line from F to J. Let k' be the unit vector associated with this line. k' belongs to the three interpretation planes and hence k' is perpendicular to P_1, P_2, and P_3. Hence these vectors are coplanar. The image frame is defined by k', P_3, and their cross product, $k' \times P_3$.

A transformation between any two frames is defined by three rotations and three translations. Such a transformation may be represented by a 4 by 4 matrix (standard homogeneous coordinates). There are nine coefficients that specify the three rotations and three coefficients that specify the three translations. Next we determine the coefficients of $\mathbf{A}_1$ and $\mathbf{A}_2$.

2.1. *The Matrix* $\mathbf{A}_1$

In order to determine the coefficients of this matrix one has to express k', P_3, and $k' \times P_3$ in the camera frame. We have:

$$k' = \frac{\mathbf{FJ}}{\|\mathbf{FJ}\|}$$

$$P_3 = \frac{l_3 \times k'}{\|l_3 \times k'\|} \qquad (4)$$

$$k' \times P_3 = \frac{k' \times (l_3 \times k')}{\|l_3 \times k'\|} = \frac{l_3 - (k' \cdot l_3)k'}{\|l_3 \times k'\|}.$$

The translational parameters are given by the coordinates of J in the camera frame.

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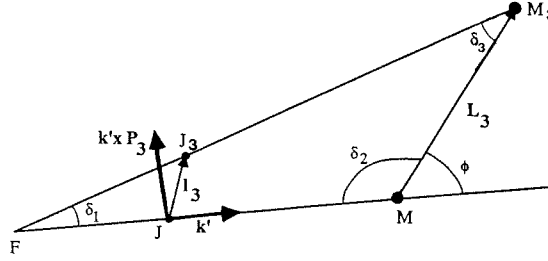
FIG. 5. The interpretation plane associated with F and l_3.

Matrix $\mathbf{A}_1$ is

$$
\mathbf{A}_1 = \begin{pmatrix}
k'_x & (P_3)_x & (k' \times P_3)_x & (\mathbf{FJ})_x \\
k'_y & (P_3)_y & (k' \times P_3)_y & (\mathbf{FJ})_y \\
k'_z & (P_3)_z & (k' \times P_3)_z & (\mathbf{FJ})_z \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{5}
$$

2.2. The Matrix $\mathbf{A}_2$

We recall that $\mathbf{A}_2$ is the transformation matrix from the object frame (defined by L_3, P_3, and $L_3 \times P_3$) to the image frame (defined by k', P_3, and $k' \times P_3$). From Fig. 5 which shows the interpretation plane associated with F and l_3 it is easy to derive an expression for L_3:

$$
L_3 = \cos \phi \, k' + \sin \phi \, k' \times P_3 \qquad \text{with the constraint } 0 < \phi < \pi. \tag{6}
$$

The rotational coefficients of $\mathbf{A}_2$ are the coordinates of L_3 in the image centered frame, i.e., Eq. (6), the coordinates of P_3, i.e., 0, 1, and 0, and the coordinates of $L_3 \times P_3$. The translational coefficients are the coordinates of the vector $\mathbf{JM}$, e.g., Fig. 4. Since the direction of this last vector is the direction of k', the matrix $\mathbf{A}_2$ is

$$
\mathbf{A}_2 = \begin{pmatrix}
\cos \phi & 0 & -\sin \phi & d_x \\
0 & 1 & 0 & 0 \\
\sin \phi & 0 & \cos \phi & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}. \tag{7}
$$

The perspective 4-point problem is reduced now to the problem of determining values for θ (present in Eqs. (2) and (3)), ϕ, and d_x. Next we derive closed form expressions for these three unknowns.

2.3. Analytic Expressions for θ and ϕ

Two more geometric constraints are available: L_1 belongs to the interpretation plane associated with l_1. Hence L_1 is orthogonal to P_1:

$$
L_1 \cdot P_1 = 0. \tag{8}
$$

Similarly, we have

$$L_2 \cdot P_2 = 0. \tag{9}$$

We express L_1, L_2, P_1, and P_2 in the image coordinate frame. By applying the transformation given by (7) to L_1 and L_2 (which are given by Eqs. (2) and (3)) we obtain (the k'-components are not relevant for this computation):

$$L_1 = (\quad)k' + \cos\alpha_1 \cos(\beta + \theta)P_3$$
$$+ (\sin\phi\sin\alpha_1 - \cos\phi\cos\alpha_1\sin(\beta + \theta))k' \times P_3 \tag{10}$$
$$L_2 = (\quad)k' + \cos\alpha_2 \cos(\beta - \theta)P_3$$
$$+ (\sin\phi\sin\alpha_2 + \cos\phi\cos\alpha_2\sin(\beta - \theta))k' \times P_3. \tag{11}$$

We have already noticed that P_1 and P_2 are perpendicular to k'. We have

$$P_1 = \cos\gamma_1 P_3 + \sin\gamma_1 k' \times P_3 \tag{12}$$
$$P_2 = \cos\gamma_2 P_3 + \sin\gamma_2 k' \times P_3. \tag{13}$$

Where γ_1 and γ_2 are given by (the unit vector normal to any interpretation plane, P_i can be determined using Eq. (4)):

$$\cos\gamma_1 = P_1 \cdot P_3 \tag{14}$$
$$\sin\gamma_1 = \qquad P_1 \cdot (k' \times P_3) \tag{15}$$
$$\cos\gamma_2 = P_2 \cdot P_3 \tag{16}$$
$$\sin\gamma_2 = \qquad P_2 \cdot (k' \times P_3) \tag{17}$$

We inject the expressions of L_1, L_2, P_1, and P_2 in Eqs. (8) and (9) and obtain

$$(\cos\gamma_1\cos\beta - \sin\gamma_1\sin\beta\cos\phi)\cos\theta$$
$$+ (-\cos\gamma_1\sin\beta - \sin\gamma_1\cos\beta\cos\phi)\sin\theta = -\sin\gamma_1\sin\phi\tan\alpha_1$$
$$(\cos\gamma_2\cos\beta + \sin\gamma_2\sin\beta\cos\phi)\cos\theta$$
$$+ (\cos\gamma_2\sin\beta - \sin\gamma_2\cos\beta\cos\phi)\sin\theta = -\sin\gamma_2\sin\phi\tan\alpha_2.$$

We determine $\sin\theta$ and $\cos\theta$ as a function of ϕ:

$$\cos\theta = \frac{\sin\phi}{D}(+K_1\cos\phi + K_2) \tag{18}$$

$$\sin\theta = \frac{\sin\phi}{D}(K_3\cos\phi + K_4) \tag{19}$$

with

$$D = K_5 - K_6 \cos^2 \phi + K_7 \cos \phi$$

$$K_1 = \sin \gamma_1 \sin \gamma_2 \cos \beta (\tan \alpha_1 - \tan \alpha_2)$$

$$K_2 = (-\sin \gamma_1 \cos \gamma_2 \tan \alpha_1 + \cos \gamma_1 \sin \gamma_2 \tan \alpha_2) \sin \beta$$

$$K_3 = \sin \gamma_1 \sin \gamma_2 \sin \beta (\tan \alpha_1 - \tan \alpha_2)$$

$$K_4 = (\sin \gamma_1 \cos \gamma_2 \tan \alpha_1 - \cos \gamma_1 \sin \gamma_2 \tan \alpha_2) \cos \beta$$

$$K_5 = \cos \gamma_1 \cos \gamma_2 \sin 2\beta$$

$$K_6 = \sin \gamma_1 \sin \gamma_2 \sin 2\beta$$

$$K_7 = \sin(\gamma_1 + \gamma_2) \cos 2\beta.$$

Finally, using the constraint $\cos^2 \theta + \sin^2 \theta = 1$, we obtain

$$I_1 \cos^4 \phi + I_2 \cos^3 \phi + I_3 \cos^2 \phi + I_4 \cos \phi + I_5 = 0 \qquad (20)$$

with

$$I_1 = K_1^2 + K_3^2 + K_6^2$$

$$I_2 = 2(K_1 K_2 + K_3 K_4 + K_6 K_7)$$

$$I_3 = K_2^2 + K_4^2 - K_1^2 - K_3^2 + K_7^2 + 2K_5 K_6$$

$$I_4 = -2(K_1 K_2 + K_3 K_4 - K_5 K_7)$$

$$I_5 = K_5^2 - K_2^2 - K_4^2.$$

The roots of Eq. (20) can be found in closed form or by an iterative method. For its real roots verifying the obvious constraint $|\cos \phi| < 1$, we can compute θ through the formulas given by Eqs. (18) and (19). With the values of ϕ and θ thus obtained we can compute L_1, L_2, and L_3 using Eqs. (10), (11), and (6). The following constraint guarantees that L_i lies in between **FM** and **FM**$_i$ (Fig. 5):

$$L_i \cdot (k' \times P_i) > 0 \qquad \text{for } i = 1, 2, 3. \qquad (21)$$

All these constraints allow us to eliminate roots which do not correspond to an admissible geometric configuration.

2.4. An Expression for d_x

From Fig. 5 it is easy to determine the length of **FM**. We have

$$\|\mathbf{MM}_3\| = \|\mathbf{FM}\| \cos \delta_2 + \|\mathbf{FM}_3\| \cos \delta_3 \qquad (22)$$

$$\|\mathbf{FM}\| \sin \delta_2 = \|\mathbf{FM}_3\| \sin \delta_3. \qquad (23)$$

From these equations we obtain

$$\|\mathbf{FM}\| = \|\mathbf{MM}_3\| \frac{\sin \delta_3}{\sin(\delta_2 + \delta_3)}. \tag{24}$$

We also have

$$\sin \delta_3 = \frac{\|\mathbf{FJ}_3 \times \mathbf{MM}_3\|}{\|\mathbf{FJ}_3\| \|\mathbf{MM}_3\|} \tag{25}$$

$$\sin(\delta_2 + \delta_3) = \sin \delta_1 = \frac{\|\mathbf{FJ} \times \mathbf{FJ}_3\|}{\|\mathbf{FJ}_3\| \|\mathbf{FJ}\|}. \tag{26}$$

Finally, we obtain

$$\|\mathbf{FM}\| = \frac{\|\mathbf{FJ}\| \|\mathbf{FJ}_3 \times \mathbf{MM}_3\|}{\|\mathbf{FJ} \times \mathbf{FJ}_3\|}. \tag{27}$$

The translation vector d_x is

$$d_x = \|\mathbf{JM}\| = \|\mathbf{FM}\| - \|\mathbf{FJ}\|. \tag{28}$$

3. SPECIAL CONFIGURATIONS

In the previous section we derived an analytical solution for the perspective 4-point problem in the general case. It is interesting to study some particular configurations associated with these four points.

3.1. Four Coplanar Points

In this section we show that the general formulation applies to four coplanar points. This situation corresponds to $2\beta = (\beta + \theta) + (\beta - \theta) = \pi$, e.g., Fig. 3. In this case we obtain

$$K_1 = K_4 = K_5 = K_6 = 0$$
$$I_2 = I_4 = 0$$
$$I_1 = K_3^2$$
$$I_3 = K_2^2 - K_3^2 + K_7^2$$
$$I_5 = -K_2^2.$$

The solution is given by the equation

$$I_1 \cos^4 \phi + I_3 \cos^2 \phi + I_5 = 0. \tag{29}$$

It is worth noticing that the discriminant of this equation is always positive:

$$\Delta = I_3^2 - 4I_1 I_5 = \left(K_2^2 - K_3^2 + K_7^2 \right)^2 + 4K_3^2 K_2^2.$$

3.2. *Three Colinear Image Points*

Another particular situation is due to an accidental alignment: The image projections of three among the four points are colinear. Let us suppose, for instance, that the image points J, J_1, and J_3 are colinear. In this case the interpretation planes P_1 and P_3 are identical. Hence we have: $\cos \gamma_1 = 1$ and $\sin \gamma_1 = 0$. We obtain

$$K_1 = K_3 = K_6 = 0$$
$$I_1 = I_2 = 0$$
$$I_3 = K_2^2 + K_4^2 - K_7^2$$
$$I_4 = -2K_5 K_7$$
$$I_5 = K_5^2 - K_2^2 - K_4^2.$$

The solution is given by

$$I_3 \cos^2 \phi + I_4 \cos \phi + I_5 = 0. \tag{30}$$

Real roots exist for this equation if and only if its discriminant is positive:

$$\Delta = \sin^2 \gamma_2 \left(\tan^2 \alpha_2 - \cos^2 2\beta \right) - \cos^2 \gamma_2 \geq 0.$$

The accidental alignment described here may correspond to a 3-line spatial vertex being projected onto the image as a T-junction. The result of this section is that such a T-junction may have a unique 3-dimensional interpretation.

3.3. *A Right Vertex*

Another particular case occurs when the four points form a right vertex, i.e., L_1, L_2, and L_3 are mutually orthogonal:

$$L_i \cdot L_j = 0 \qquad \text{for } i \neq j. \tag{31}$$

In this case a simpler solution than the general case can be derived. Notice first that Eq. (6) can also be written for L_1 and L_2:

$$L_i = \cos \phi_i k' + \sin \phi_i k' \times P_3 \qquad \text{with } i = 1, 2, 3. \tag{32}$$

For notation homogeneity ϕ is replaced by ϕ_3. We obtain

$$L_i \cdot L_j = \cos \phi_i \cos \phi_j + \sin \phi_i \sin \phi_j \left(P_i \cdot P_j \right) \qquad \text{for } i \neq j. \tag{33}$$

We have already mentioned that P_1, P_2, and P_3 are coplanar (they are all orthogonal to k'). Hence the three dot products $P_i \cdot P_j$ cannot be simultaneously null. There are three possible situations:

1. $P_1 \cdot P_2 = 0$, $P_1 \cdot P_3 \neq 0$, and $P_2 \cdot P_3 \neq 0$. We obtain $\phi_1 = \phi_2 = 0$ which is impossible because it corresponds to two space points which project onto the image at the same location, e.g., Fig. 5;

2. $P_1 \cdot P_2 = 0$, $P_1 \cdot P_3 = 0$, and $P_2 \cdot P_3 \neq 0$. P_1 is perpendicular to both P_2 and P_3, hence they are colinear, $|P_2 \cdot P_3| = 1$. We obtain

$$\cos \phi_1 \cos \phi_2 = 0 \qquad (34)$$

$$\cos \phi_1 \cos \phi_3 = 0 \qquad (35)$$

$$\sin \phi_2 \sin \phi_3 \pm \cos \phi_2 \cos \phi_3 = 0, \qquad (36)$$

which gives the solutions $\phi_1 = \pi/2$ and $\phi_2 \pm \phi_3 = \pi/2$, and

3. $P_1 \cdot P_2 \neq 0$, $P_1 \cdot P_3 \neq 0$, and $P_2 \cdot P_3 \neq 0$. We obtain

$$\tan \phi_2 = \tan \phi_3 \frac{P_1 \cdot P_3}{P_1 \cdot P_2} \qquad (37)$$

$$\tan \phi_1 = \tan \phi_3 \frac{P_2 \cdot P_3}{P_1 \cdot P_2} \qquad (38)$$

$$\tan^2 \phi_3 = -\frac{P_1 \cdot P_2}{(P_1 \cdot P_3)(P_2 \cdot P_3)}. \qquad (39)$$

In this case solutions exist under the constraint: $(P_1 \cdot P_2)(P_1 \cdot P_3)(P_2 \cdot P_3) < 0$.

4. DISCUSSION

In this paper we derived an analytic solution for computing the exterior camera parameters from four correspondence points in general positions. This solution is of the same complexity as for three points and is particularly simple for such configurations as four coplanar points or four points forming a right vertex. Such an analytic formulation allows fast numerical computation which is desirable in many applications such as on line calibration (hand/eye coordination, navigation) and/or object recognition and positioning from a single view.

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